Generalized scale-invariant solutions to the two-dimensional stationary Navier–Stokes equations

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2015

New explicit solutions to the incompressible Navier–Stokes equations in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ are determined, which generalize the scale-invariant solutions found by Hamel. These new solutions are invariant under a particular combination of the scaling and rotational symmetries. They are the only solutions invariant under this new symmetry in the same way as the Hamel solutions are the only scale-invariant solutions. While the Hamel solutions are parameterized by a discrete parameter n, the flux Φ, and an angle θ_0, the new solutions generalize the Hamel solutions by introducing an additional parameter κ which produces a rotation. The new solutions decay like $|\boldsymbol{x}|^{-1}$ as the Hamel solutions and exhibit spiral behavior. The new variety of asymptotes induced by the existence of these solutions further emphasizes the difficulties faced when trying to establish the asymptotic behavior of the Navier–Stokes equations in a two-dimensional exterior domain or in the whole plane.

Keywords Navier–Stokes equations, exact solutions, spirals, fluid-structure interactions
MSC classes 76D05, 76D03, 76U05, 35Q30, 74F10
Reference SIAM Journal on Mathematical Analysis **47** (1), 955–968, 2015
DOI 10.1137/140963030

1. Introduction

We study a new special class of solutions to the stationary incompressible Navier–Stokes equations in $\Omega = \mathbb{R}^2 \setminus \{\mathbf{0}\}$,

$$\Delta \boldsymbol{u} - \boldsymbol{\nabla} p = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}\,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,, \qquad \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{u} = \mathbf{0}\,. \tag{1}$$

An important parameter which labels the solutions of this system is the flux,

$$\Phi = \int_\gamma \boldsymbol{u} \cdot \boldsymbol{n}\,, \tag{2}$$

which is independent of the choice of any simple closed curve γ encircling the origin. The equations (1) are invariant under two types of symmetries: the rotations around the origin $\boldsymbol{u}(\boldsymbol{x}) \mapsto \mathbf{R}^{-1}\boldsymbol{u}(\mathbf{R}\boldsymbol{x})$, with $\mathbf{R} \in \mathrm{SO}(2)$, and the scaling $\boldsymbol{u}(\boldsymbol{x}) \mapsto \mathrm{e}^{\lambda}\boldsymbol{u}(\mathrm{e}^{\lambda}\boldsymbol{x})$, with $\lambda \in \mathbb{R}$. The solutions that are invariant under these symmetries play a particular role (Wang, 1991, pp. 168–173) in the asymptotic behavior of the Navier–Stokes equations, as explained later. Šverák (2011) studied in detail the scale-invariant solutions of the Navier–Stokes equations in dimension $d \geq 2$. In three dimensions, the only scale-invariant solutions are the Landau (1944) solutions, which decay like $|\boldsymbol{x}|^{-1}$ and are labeled by a vector in $\mathbb{R}^3$ whose norm determines the force acting on the fluid. In two dimensions, Šverák (2011, §5) showed that the only scale-invariant solutions of (1) are the Hamel (1917, §6) solutions. The Hamel solutions are characterized by the flux Φ and a discrete parameter $n \in \mathbb{N}$, with an additional parameter μ for $n = 0$. In polar coordinates (r, θ) they are given for $n = 0$ by

$$\boldsymbol{u}_{\Phi,0} = \frac{\Phi}{2\pi r}\boldsymbol{e}_r + \frac{\mu}{r}\boldsymbol{e}_\theta \,, \tag{3}$$

where $\mu \in \mathbb{R}$ is an additional parameter, and by

$$\boldsymbol{u}_{\Phi,n} = \frac{-1}{r}\varphi(\theta + \theta_0)\boldsymbol{e}_r \,, \tag{4}$$

for $n \in \mathbb{N}^*$ and $4 + \frac{\Phi}{\pi} \leq n^2$, where φ is a $\frac{2\pi}{n}$-periodic function determined by n and Φ, and θ_0 is an angle that can be chosen arbitrarily. In view of their special form these solutions are scale-invariant, *i.e.* $\boldsymbol{u}(\boldsymbol{x}) = \mathrm{e}^{\lambda}\boldsymbol{u}(\mathrm{e}^{\lambda}\boldsymbol{x})$. Moreover, it is interesting to note that in the case $n = 0$, Hamel (1917, §11) found one more free parameter $A \in \mathbb{R}$ since

$$\boldsymbol{u}_{\Phi,\mu,A} = \frac{\Phi}{2\pi r}\boldsymbol{e}_r + \left(\frac{\mu}{r} + A\gamma(r)\right)\boldsymbol{e}_\theta \,, \tag{5}$$

where

$$\gamma(r) = \begin{cases} r^{1+\frac{\Phi}{2\pi}} \,, & \Phi \neq -4\pi \,, \\ \frac{\log r}{r} \,, & \Phi = -4\pi \,, \end{cases}$$

is an exact solution of (1) provided $A = 0$ for $\Phi > -2\pi$, otherwise this solution does not decay at infinity. This solution is rotational-invariant but not scale-invariant for $A \neq 0$ and is bounded by r^{-1} at infinity only for $\Phi < -4\pi$.

In what follows we look for solutions invariant under combinations of the scaling and rotational symmetries. Together these symmetries form a two-dimensional group G isomorphic to the additive group $\mathbb{R} \times S^1$. The action of $(\lambda, \vartheta) \in G$ on the velocity field is given by $\boldsymbol{u}(\boldsymbol{x}) \mapsto \mathrm{e}^{\lambda}\mathbf{R}_{\vartheta}^{-1}\boldsymbol{u}(\mathrm{e}^{\lambda}\mathbf{R}_{\vartheta}\boldsymbol{x})$, where $\mathbf{R}_{\vartheta}$ is the rotation matrix of angle ϑ. The one-dimensional subgroups of G are given by

$$G_\kappa = \{(\lambda, \vartheta) \in G \;:\; \kappa\lambda + \vartheta = 0\} \,, \qquad G_\infty = \{(\lambda, \vartheta) \in G \;:\; \lambda = 0\} \,,$$

where $\kappa \in \mathbb{R}$ is a parameter. The aim of this paper is to determine all solutions of (1) that are invariant under any one-dimensional subgroup of G and to discuss their implications. The subgroup G_0 corresponds to scale-invariant solutions and the subgroup G_∞ to rotation-invariant solutions. We say that a solution $\boldsymbol{u}$ of the Navier–Stokes equations (1) is scale-invariant up to a rotation if the solution is invariant under the subgroup G_κ for $\kappa \neq 0$, *i.e.*

$$\boldsymbol{u}(\boldsymbol{x}) = \mathrm{e}^{\lambda}\mathbf{R}_{-\kappa\lambda}^{-1}\boldsymbol{u}(\mathrm{e}^{\lambda}\mathbf{R}_{-\kappa\lambda}\boldsymbol{x}) \,, \tag{6}$$

for all $\lambda \in \mathbb{R}$.

Our main results are the following:

Theorem 1. *For all $\Phi, \kappa \in \mathbb{R}$, and $n \in \mathbb{N}^*$ satisfying*

$$\frac{4 + \frac{\Phi}{\pi}}{1 + \kappa^2} \leq n^2 \,,$$

there exists a $\frac{2\pi}{n}$-periodic function φ depending on n, Φ, and κ such that for any $\theta_0 \in \mathbb{R}$,

$$\boldsymbol{u}_{\Phi,n,\kappa} = \frac{1}{r}\Big[-\varphi(\theta + \kappa\log r + \theta_0)\boldsymbol{e}_r + \kappa\left(\varphi(\theta + \kappa\log r + \theta_0) - 4\right)\boldsymbol{e}_\theta\Big] \tag{7}$$

and the associated pressure (13) *satisfy the Navier–Stokes equations* (1)*. These solutions are invariant under the subgroup* $\boldsymbol{G}_\kappa$ *and have flux* Φ*. Moreover, any solution of the Navier–Stokes equations* (1) *which is invariant under the subgroup* $\boldsymbol{G}_\kappa$ *is equal either to one of the exact solutions* $\boldsymbol{u}_{\Phi,n,\kappa}$ *for an angle* θ_0 *or to a Hamel solution* $\boldsymbol{u}_{\Phi,0}$ *defined by* (3) *with* $n = 0$ *for some* $\mu \in \mathbb{R}$*.*

Theorem 2. *For all* $\Phi, \mu, A \in \mathbb{R}$ *satisfying* $A = 0$ *if* $\Phi > -2\pi$*, the Hamel solution* $\boldsymbol{u}_{\Phi,\mu,A}$ *defined by* (5) *and the associated pressure* (23) *satisfy the Navier–Stokes equations* (1)*. These solutions are the only solutions invariant under the subgroup* $\boldsymbol{G}_\infty$*, that is, under rotations.*

Remark 3. The solutions that are scale-invariant and rotation-invariant, *i.e.* under the whole group $\boldsymbol{G}$, are given by (3) or by (5) with $A = 0$.

Remark 4. The ansatz for spiral solutions made by Hamel (1917, §9) does not allow solutions in the whole plane with streamlines that are logarithmic spirals. The solutions with logarithmic spirals that he found are possible only between two walls of logarithmic shape. The existence of solutions presented here shows that the intuition of Hamel to look for non-harmonic functions in the plane having streamlines that are spirals, was nevertheless basically correct.

The expression (7) is a solution of the Navier–Stokes equations in $\mathbb{R}^2 \setminus \{\boldsymbol{0}\}$, but due to the behavior near the origin like r^{-1}, the non-linear term $\boldsymbol{u} \cdot \boldsymbol{\nabla u}$, even when written as $\boldsymbol{\nabla} \cdot (\boldsymbol{u} \otimes \boldsymbol{u})$, has no immediate distributional meaning in $\mathbb{R}^2$. This is in contrast to the three-dimensional case where the non-linear term of a scale-invariant solution is a distribution even if $\boldsymbol{u}$ diverges likes r^{-1} at the origin. One can nevertheless always construct a solution to the Navier–Stokes equations in $\mathbb{R}^2$ by truncating one of the exact solutions near the origin and defining the source term by the truncation error. The force and the torque of a solution $\boldsymbol{u}$ are given for any curve γ encircling the origin, by

$$\boldsymbol{F} = \int_\gamma \mathbf{T}\boldsymbol{n} \,, \qquad\qquad M = \int_\gamma \boldsymbol{x} \wedge \mathbf{T}\boldsymbol{n} \,,$$

where $\mathbf{T}$ is the stress tensor including the convective part, $\mathbf{T} = \boldsymbol{u} \otimes \boldsymbol{u} + p - \boldsymbol{\nabla u} - (\boldsymbol{\nabla u})^T$. By taking for γ a circle whose radius goes to infinity, the force is zero, $\boldsymbol{F} = \boldsymbol{0}$. By taking for simplicity the circle of radius one, the torque is

$$M = \kappa \left(16\pi + 6\Phi + \int_{-\pi}^{+\pi} \varphi^2(\theta)\, \mathrm{d}\theta \right) .$$

The study of scale-invariant solutions has proved to be of great importance, in particular for the determination of the asymptotic behavior of the stationary Navier–Stokes equations in two or three dimensions. The stationary and incompressible Navier–Stokes equations in the exterior domain $\Omega = \mathbb{R}^2 \setminus B$ of a compact, connected set B are

$$\begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{\nabla} p &= \boldsymbol{u} \cdot \boldsymbol{\nabla u} \,, & \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \,, \\ \boldsymbol{u}|_{\partial B} &= \boldsymbol{u}^* \,, & \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} &= \boldsymbol{0} \,, \end{aligned} \tag{8a}$$

where $\boldsymbol{u}^*$ is any smooth boundary condition with no net flux,

$$\int_{\partial B} \boldsymbol{u}^* \cdot \boldsymbol{n} = 0 \,. \tag{8b}$$

Problem (8) is closely related to the one of the incompressible Navier–Stokes equations in $\mathbb{R}^2$,

$$\Delta\boldsymbol{u} - \boldsymbol{\nabla} p - \boldsymbol{u} \cdot \boldsymbol{\nabla u} = \boldsymbol{f} \,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \qquad \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} = \boldsymbol{0} \,, \tag{9}$$

where $\boldsymbol{f}$ is a smooth function of compact support. We remark that the problems (8) and (9) are very similar on a formal level: any solution of (8) defines a solution of (9) on the exterior of the support of $\boldsymbol{f}$, and conversely any solution of (8) can be truncated in order to obtain a solution of (9). In three dimensions, Nazarov & Pileckas (1999, 2000) proved that the asymptotic behavior of solutions of (8) is a scale-invariant solution. Then Korolev & Šverák (2011) simplified the proof by showing directly that, in this case, the Landau solution is the correct asymptotic behavior of any solution bounded by $(1+|\boldsymbol{x}|)^{-1}$. In two dimensions, existence of solutions to (8) or (9) are not known in general (Galdi, 2004; Guillod & Wittwer, 2015), even for small data. The difference between two and three dimensions is essentially that in three dimensions the compatibility condition of the Stokes approximation to decay faster than r^{-1} at infinity corresponds to the force and can be lifted by the Landau solutions which are exact solutions of (9) with $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{b}\delta(\boldsymbol{x})$, where $\boldsymbol{b} \in \mathbb{R}^3$ is the net force and δ is the Dirac distribution. In two dimensions and in the case where $\boldsymbol{f}$ has non-zero mean, Guillod & Wittwer (2015) showed by physical arguments and detailed numerical verification that the velocity has to decay like $r^{-1/3}$ at infinity. In the case where $\boldsymbol{f}$ has zero mean, one would guess by analogy with the three-dimensional case that the asymptotic behavior should be a scale-invariant solution. However, Šverák (2011, §5) shows that one cannot prove this by using perturbation techniques based on the Stokes approximation, and even together with the newly discovered solutions, we do not appear to be able to parameterize the general asymptotic behavior in the case where $\boldsymbol{f}$ has zero mean. The intuitive reason for this is the fact that the Stokes approximation has two compatibility conditions if we require the solution to decay faster than r^{-1} at infinity: one of them might be lifted by adjusting the parameter κ of the new solutions, but we do not have sufficient parameters to lift also the other compatibility condition. We believe that the newly discovered solutions are a special case of a more general family of solutions, yet to be discovered, with one more parameter, corresponding to the general asymptotic behavior in the case where $\boldsymbol{f}$ has zero mean.

The paper in organized as follows. We first prove that the solutions which are scale-invariant up to a rotation are given explicitly in terms of a 2π-periodic function φ satisfying an ordinary differential equation, and then we solve this differential equation by using elliptic functions. We then determine the rotation-invariant solutions explicitly. Finally, we represent the solutions graphically, analyze the solutions having small amplitude, and discuss the implications for the solutions of Navier–Stokes equations.

2. Scale-invariant solutions up to a rotation

In this section, we prove theorem 1 by analyzing the solutions of (1) that are invariant under the subgroup G_κ for $\kappa \in \mathbb{R}$, that is, that are either scale-invariant (for $\kappa = 0$) or scale-invariant up to a rotation (for $\kappa \neq 0$).

2.1. Reduction to an ordinary differential equation

For $\kappa \in \mathbb{R}$, we consider a solution $\boldsymbol{u}$ which is invariant under G_κ, as defined by (6). In polar coordinates (r,θ) this symmetry is more easily expressed,

$$u_r(r,\theta) = \mathrm{e}^{\lambda} u_r(\mathrm{e}^{\lambda} r, \theta - \kappa\lambda)\,, \qquad u_\theta(r,\theta) = \mathrm{e}^{\lambda} u_\theta(\mathrm{e}^{\lambda} r, \theta - \kappa\lambda)\,.$$

Therefore, by setting $\lambda = -\log r$, u_r and u_θ are characterized by their values on S^1,

$$u_r(r,\theta) = \frac{1}{r}\varphi_r(\theta + \kappa\log r)\,, \qquad u_\theta(r,\theta) = \frac{1}{r}\varphi_\theta(\theta + \kappa\log r)\,,$$

where $\varphi_i(\theta) = u_i(1,\theta)$ for $i \in \{r,\theta\}$. The divergence of the vector field $\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{1}{r^2}\left[\varphi_\theta'(\varrho) + \kappa\varphi_r'(\varrho)\right]\,,$$

where

$$\varrho = \theta + \kappa\log r\,. \tag{10}$$

The requirement of $\boldsymbol{u}$ to be divergence free therefore implies that

$$\varphi_\theta(\varrho) = \mu - \kappa\varphi_r(\varrho)\,,$$

where μ is a real constant. Consequently, a divergence-free vector field satisfies the symmetry (6), if and only if it has the form

$$\boldsymbol{u} = \frac{1}{r}\left[-\varphi(\varrho)\boldsymbol{e}_r + (\mu + \kappa\varphi(\varrho))\,\boldsymbol{e}_\theta\right]\,, \qquad \varrho = \theta + \kappa\log r\,, \tag{11}$$

where φ is a 2π-periodic function. The corresponding stream function ψ, defined such that $\boldsymbol{u} = \boldsymbol{\nabla}\wedge\psi$, is

$$\psi(r,\theta) = \mu\log r + \Gamma(\varrho)\,,$$

where Γ is an antiderivative of φ.

We now determine the ordinary differential equation which φ has to satisfy in order for $\boldsymbol{u}$ to be an exact solution of (1). The vorticity is

$$\omega = \frac{1+\kappa^2}{r^2}\varphi'(\varrho)\,,$$

and the vorticity equation

$$\Delta\omega = \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega$$

becomes, after an explicit integration, the ordinary differential equation

$$\left(1+\kappa^2\right)\varphi''(\varrho) - (\mu + 4\kappa)\,\varphi'(\varrho) + 4\varphi(\varrho) = \varphi(\varrho)^2 - C\,, \tag{12}$$

where $C \in \mathbb{R}$ is a constant related to certain averages of φ. By integrating the Navier–Stokes equations we can construct the pressure,

$$p = \frac{1}{r^2}\left[\kappa\left(1+\kappa^2\right)\varphi'(\varrho) - \left(2\left(1+\kappa^2\right) + \kappa\mu\right)\varphi(\varrho)\right] - \frac{1}{2r^2}\left[\mu^2 + C\left(1+\kappa^2\right)\right]\,. \tag{13}$$

This shows that the only solutions of (1) which are scale-invariant up to a rotation are given by (11) and (13), where φ is a 2π-periodic function satisfying (12). The differential equation (12) is analogous to the one describing the motion of a particle in a potential undergoing friction, and in order to obtain periodic solutions, the damping term has to vanish, *i.e.* $\mu + 4\kappa = 0$. So we finally end up with a differential equation with two parameters: κ and C. In the next section we find the periodic solutions of this differential equation.

2.2. Resolution of the ordinary differential equation

Finding the solutions of the ordinary differential equation (12) is rather standard (Rosenhead, 1940; Šverák, 2011, Theorem 2). The change of variable $\varrho \mapsto \left(1+\kappa^2\right)^{-1/2}\varrho$ transforms this differential equation to the same one with $\kappa = 0$, but the 2π-periodicity condition on φ is not preserved. Therefore, we cannot use directly the existing results on the scale-invariant solutions to treat the general case $\kappa \neq 0$. The differential equation (12) is clearly invariant under the translation $\varrho \mapsto \varrho + \theta_0$, so we do not keep track of this trivial symmetry and fix the origin later on in a convenient way. The trivial solutions where φ is constant are not included in this analysis, since they correspond to the Hamel solutions (3). As explained above, in order to obtain periodic solutions we have to take $\mu + 4\kappa = 0$, and therefore the ordinary differential equation (12) can be written as the differential equation describing a free particle in a potential,

$$\varphi'' = -V'(\varphi)\,, \qquad V(\varphi) = \frac{1}{1+\kappa^2}\left[C\varphi + 2\varphi^2 - \frac{\varphi^3}{3}\right]\,.$$

The energy is conserved so

$$E = \frac{1}{2}\left(\varphi'\right)^2 + V(\varphi)\,, \qquad\qquad \varphi' = \pm\sqrt{2E - 2V(\varphi)}\,.$$

Since we look for non-trivial periodic solutions, the potential has to have a minimum, so $C > -4$, and the energy has to be between the maximum and the minimum admissible values,

$$\left|\left(1+\kappa^2\right)E - 2C - \frac{16}{3}\right| < \frac{2}{3}(C+4)^{3/2}\,.$$

These two conditions imply that the polynomial $2E - 2V(\varphi)$ has three distinct real roots, $\varphi_1 < \varphi_2 < \varphi_3$, and by Vieta's formulas,

$$\varphi_1 + \varphi_2 + \varphi_3 = 6\,. \tag{14}$$

Therefore,

$$2E - 2V(\varphi) = \frac{2}{3\left(1+\kappa^2\right)}\left(\varphi-\varphi_1\right)\left(\varphi-\varphi_2\right)\left(\varphi-\varphi_3\right)\,,$$

and the solution is given in term of the incomplete elliptic function of the first kind F,

$$\varrho(\varphi) = \sqrt{\frac{3}{2}}\sqrt{1+\kappa^2}\int_{\varphi_1}^{\varphi}\frac{\mathrm{d}\varphi}{\sqrt{\left(\varphi-\varphi_1\right)\left(\varphi-\varphi_2\right)\left(\varphi-\varphi_3\right)}} = \sqrt{6}\frac{\sqrt{1+\kappa^2}}{\sqrt{\varphi_3-\varphi_1}}F\left(\sqrt{\frac{\varphi-\varphi_1}{\varphi_2-\varphi_1}};\alpha\right)\,, \tag{15}$$

where

$$\alpha = \sqrt{\frac{\varphi_2-\varphi_1}{\varphi_3-\varphi_1}}\,. \tag{16}$$

We take the following convention for the elliptic integral F (Byrd & Friedman, 1971, #110.02):

$$F(x;\alpha) = \int_0^x \frac{\mathrm{d}t}{\sqrt{\left(1-t^2\right)\left(1-\alpha^2t^2\right)}} = \int_0^{\arcsin x}\frac{\mathrm{d}\theta}{\sqrt{1-\alpha^2\sin^2\theta}}\,.$$

The function φ is 2π-periodic if there exists $n \in \mathbb{N}^*$ such that

$$\varrho(\varphi_2) = \frac{\pi}{n}\,,$$

i.e., explicitly,

$$\frac{2\sqrt{1+\kappa^2}K(\alpha)}{\sqrt{\varphi_3-\varphi_1}} = \sqrt{\frac{2}{3}}\frac{\pi}{n}\,, \tag{17}$$

where K is the complete elliptic function of the first kind. The flux is given by

$$\Phi = \int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{n} = \int_{\partial B(\boldsymbol{0},1)}\boldsymbol{u}\cdot\boldsymbol{e}_r = -\int_{-\pi}^{+\pi}\varphi(\varrho)\,\mathrm{d}\varrho = -2n\int_0^{\pi/n}\varphi(\varrho)\,\mathrm{d}\varrho\,,$$

and, explicitly, by using the complete elliptic function of the second kind E,

$$\Phi = -n\sqrt{6}\int_{\varphi_1}^{\varphi_2}\frac{\sqrt{1+\kappa^2}\,\varphi\,\mathrm{d}\varphi}{\sqrt{\left(\varphi-\varphi_1\right)\left(\varphi-\varphi_2\right)\left(\varphi-\varphi_3\right)}} = -2\sqrt{6}n\frac{\sqrt{1+\kappa^2}}{\sqrt{\varphi_3-\varphi_1}}\left[\varphi_3K(\alpha) - \left(\varphi_3-\varphi_1\right)E(\alpha)\right]\,. \tag{18}$$

The conditions (14), (17), and (18) reduce to

$$H(\alpha) = \frac{1}{n^2\left(1+\kappa^2\right)}\left(\pi^2 + \frac{\pi\Phi}{4}\right), \qquad H(\alpha) = \left[\left(\alpha^2-2\right)K(\alpha) + 3E(\alpha)\right]K(\alpha)\,. \tag{19}$$

The function $H : [0,1) \to \left(-\infty, \frac{\pi^2}{4}\right]$ is bijective (see appendix §A), so this equation has a unique solution $\alpha_n > 0$ for each $n \in \mathbb{N}^*$ satisfying

$$\frac{4+\frac{\Phi}{\pi}}{1+\kappa^2} \leq n^2\,. \tag{20}$$

Finally, (15) can be inverted and the solution is explicitly given by

$$\varphi(\varrho) = \varphi_1 + \left(\varphi_2 - \varphi_1\right)\operatorname{sn}\left(\frac{\sqrt{\varphi_3-\varphi_1}}{\sqrt{6}\sqrt{1+\kappa^2}}\varrho, \alpha\right)^2,$$

where sn is the Jacobi elliptic function. In this expression, α is determined by solving (19) with respect to κ, Φ and n inside the region defined by (20). The φ_i for $i \in \{1,2,3\}$ are determined by solving (14), (16) and (17).

3. Rotation-invariant solutions

In this section, we prove theorem 2 which states that the Hamel solutions (5) are the only rotation-invariant-solutions. We consider a solution $\boldsymbol{u}$ which is invariant under G_∞, *i.e.* under rotations. In polar coordinates, we have for such solutions

$$u_r(r,\theta) = u_r(r,\theta+\vartheta)\,, \qquad u_\theta(r,\theta) = u_\theta(r,\theta+\vartheta)\,,$$

for all $\vartheta \in \mathbb{R}$. So by setting $\vartheta = -\theta$, we see that u_r and u_θ are independent of the angle θ, and therefore this vector is divergence free if and only if it has the form

$$\boldsymbol{u} = \frac{\Phi}{2\pi r}\boldsymbol{e}_r + \rho(r)\boldsymbol{e}_\theta\,, \tag{21}$$

where ρ is a function. The corresponding stream function is given by

$$\psi = -\frac{\Phi}{2\pi}\theta + \Omega(r)\,,$$

where Ω is an antiderivative of ρ, and the vorticity is

$$\omega = \rho'(r) + \frac{\rho(r)}{r}\,.$$

After an explicit integration, the vorticity equation becomes an ordinary differential equation for ρ,

$$\rho''(r) - \left(\frac{\Phi}{2\pi} - 1\right)\frac{\rho'(r)}{r} - \left(\frac{\Phi}{2\pi} + 1\right)\frac{\rho(r)}{r^2}\,.$$

The general solution of this linear differential equation is given by

$$\rho(r) = \frac{\mu}{r} + A\gamma(r)\,, \qquad \gamma(r) = \begin{cases} r^{1+\frac{\Phi}{2\pi}}\,, & \Phi \in \mathbb{R}\setminus\{-4\pi, 0\}\,, \\ \frac{\log r}{r}\,, & \Phi = -4\pi\,, \\ r\log r\,, & \Phi = 0\,. \end{cases} \tag{22}$$

By integrating the Navier–Stokes equation (21), we recover the pressure,

$$p = -\frac{1}{2}\left(\frac{\Phi}{2\pi}\right)^2 + \Pi(r)\,, \tag{23}$$

where Π is an antiderivative of $\frac{\rho^2(r)}{r}$. This proves that the only rotation-invariant solutions of (1) are given by (21) and (23), where ρ is given by (22), and $\mu, A \in \mathbb{R}$ with the restriction $A = 0$ for $\Phi > -2\pi$.

4. Discussion of solutions

For $n \in \mathbb{N}^*$, the exact solution $\boldsymbol{u}_{\Phi,n,\kappa}$ exists provided condition (20) is satisfied. The corresponding region in the plane (κ, Φ) is represented in figure 1. Moreover, the Hamel solution $\boldsymbol{u}_{\Phi,\mu,A}$ defined by (5) with $A \neq 0$ exists for $\Phi < -2\pi$ and decays like r^{-1} at infinity if $\Phi \leq -4\pi$, and less rapidly if $-4\pi < \Phi < -2\pi$. The linearization of the Navier–Stokes equations around the harmonic function (3) with $\mu = -4\kappa$ can be solved exactly by the use of a Fourier series (Hillairet & Wittwer, 2013). Except the zero coefficient, the Fourier modes decay faster than r^{-1} provided a condition on Φ and μ holds. This condition is represented in figure 1 by a red curve, so that for all values above the curve the Fourier modes decay faster than r^{-1} at infinity. The zero-flux case was treated by Hillairet & Wittwer (2013), and they found that provided $|\kappa| > \sqrt{3}$, the Navier–Stokes equations in the exterior of a unit disk with a Dirichlet boundary condition sufficiently close to $-4\kappa\boldsymbol{e}_\theta$ admit a solution whose asymptote is given by $-4\tilde{\kappa}\boldsymbol{e}_\theta/r$, where $\tilde{\kappa}$ is close to κ.

For a given n, the solution φ has n maxima and n minima (figure 2) and the parameter κ has the effect of rotating the branches corresponding to these maxima or minima, as shown in figure 4. Small solutions of (1) are of particular interest because even in this case we don't know the existence of a solution in general. Certain large solutions of the Navier–Stokes equations are known to exhibit exotic behavior at infinity, like, for example, the Hamel solutions, which have arbitrarily slow decay to infinity and violate uniqueness (Galdi, 2011, XII.2). Small solutions of the ordinary differential equation are given by α small, and to discuss these solutions we develop $H(\alpha)$ in a series,

$$H(\alpha) = \frac{\pi^2}{4}\left(1 - \frac{3}{32}\alpha^4\right) + O(\alpha^6)\,.$$

In view of (19), small solutions with $\Phi = 0$ are possible only for $n \in \{1, 2\}$. For $n = 1$ and $\Phi = 0$, the solutions are defined for $|\kappa| \geq \sqrt{3}$, and we take for example $\kappa = \sqrt{3} + \varepsilon$, with $\varepsilon > 0$. We find by a series expansion that

$$\alpha = \left(\frac{256}{3}\right)^{1/8} \varepsilon^{1/4} + O(\varepsilon^{1/2})$$

and

$$\varphi_1 = -4\,3^{3/4}\varepsilon^{1/2} + O(\varepsilon)\,, \qquad \varphi_2 = 4\,3^{3/4}\varepsilon^{1/2} + O(\varepsilon)\,, \qquad \varphi_3 = 6 + O(\varepsilon)\,,$$

and the solution satisfies

$$\varphi(\varrho) = -4\,3^{3/4}\varepsilon^{1/2}\cos(\varrho) + O(\varepsilon)\,.$$

Since in this case $|\kappa|$ has to be large, we note that this does not produce a solution having a small velocity field, and the torque is also large, $M = 16\pi\sqrt{3} + O(\varepsilon)$. Moreover, since $\mu + 4\kappa = 0$, this corresponds to $|\mu| \geq \sqrt{48}$. This specific value is interesting since this is exactly the criterion found by Hillairet & Wittwer (2013) to obtain a solution of the Navier–Stokes equations (8) having the asymptote $\mu\boldsymbol{e}_\theta/r$ for the velocity. We note that this is not in contradiction with the existence of the exact solutions found here, because the boundary condition given by the evaluation of the exact solution for $\kappa = \sqrt{3} + \varepsilon$ is too big for the theorem of Hillairet & Wittwer (2013) to apply.

For $n = 2$, we can take $\kappa = \varepsilon$, so

$$\alpha = \left(\frac{32}{3}\right)^{1/4} \varepsilon^{1/2} + O(\varepsilon)\,,$$

and

$$\varphi_1 = -4\sqrt{6}\,\varepsilon + O(\varepsilon^2)\,, \qquad \varphi_2 = 4\sqrt{6}\,\varepsilon + O(\varepsilon^2)\,, \qquad \varphi_3 = 6 + O(\varepsilon^2)\,,$$

and the solution satisfies

$$\varphi(\varrho) = -4\sqrt{6}\,\varepsilon\cos(2\varrho) + O(\varepsilon^2)\,.$$

The torque of this solution is given by $M = 16\pi\varepsilon + O(\varepsilon^2)$.

We now discuss the consequences of the existence of the newly found solutions for the solutions of the Navier–Stokes equations (9) in $\mathbb{R}^2$. For the reasons explained in the introduction, if $\boldsymbol{f}$ has non-zero mean, the solution cannot decay like r^{-1}. Therefore, the new solutions describe, at best, the asymptotes of solutions for the case where $\boldsymbol{f}$ has zero mean. To discuss this question, we consider the Stokes approximation

$$\Delta\boldsymbol{u} - \boldsymbol{\nabla}p = \boldsymbol{f}\,,$$

with $\boldsymbol{f}$ having zero mean. The asymptotic behavior of the solution to this equation is

$$\boldsymbol{u} = \frac{-1}{4\pi r}\left[\left(\cos(2\theta)\int_{\mathbb{R}^2}\left(x_1 f_1 - x_2 f_2\right) + \sin(2\theta)\int_{\mathbb{R}^2}\left(x_1 f_2 + x_2 f_1\right)\right)\boldsymbol{e}_r + \int_{\mathbb{R}^2}(\boldsymbol{x}\wedge\boldsymbol{f})\,\boldsymbol{e}_\theta\right] + O(r^{-2})\,,$$

By rotating the coordinates system, we can always make $\int_{\mathbb{R}^2}\left(x_1 f_2 + x_2 f_1\right) = 0$, so that the Stokes approximation has two compatibility conditions: the torque $\int_{\mathbb{R}^2}(\boldsymbol{x}\wedge\boldsymbol{f})$ and $\int_{\mathbb{R}^2}\left(x_1 f_1 - x_2 f_2\right)$ which are represented in figure 3. Even if the small solution found for $n = 2$ has the appropriate form $\cos(2\varrho)$ which is similar to the $\cos(2\theta)$ of the Stokes solution, the asymptotic behavior is likely not described by this exact solution alone, because only one of the compatibility conditions can be lifted by this exact solution. In addition, we note that there are two exact solutions decaying like r^{-1} and having arbitrarily small torque: the harmonic function $\mu\boldsymbol{e}_\theta/r$ and the new solution for $n = 2$. This emphasizes the wide variety of asymptotic behavior of the solutions to (9) with small data, since by truncating one of these solutions near the origin we obtain an exact solution in $\mathbb{R}^2$ with a certain small source term. In fact, numerical studies make us believe that even if the source term has zero mean, the solution of (9) is in general not bounded by r^{-1}.

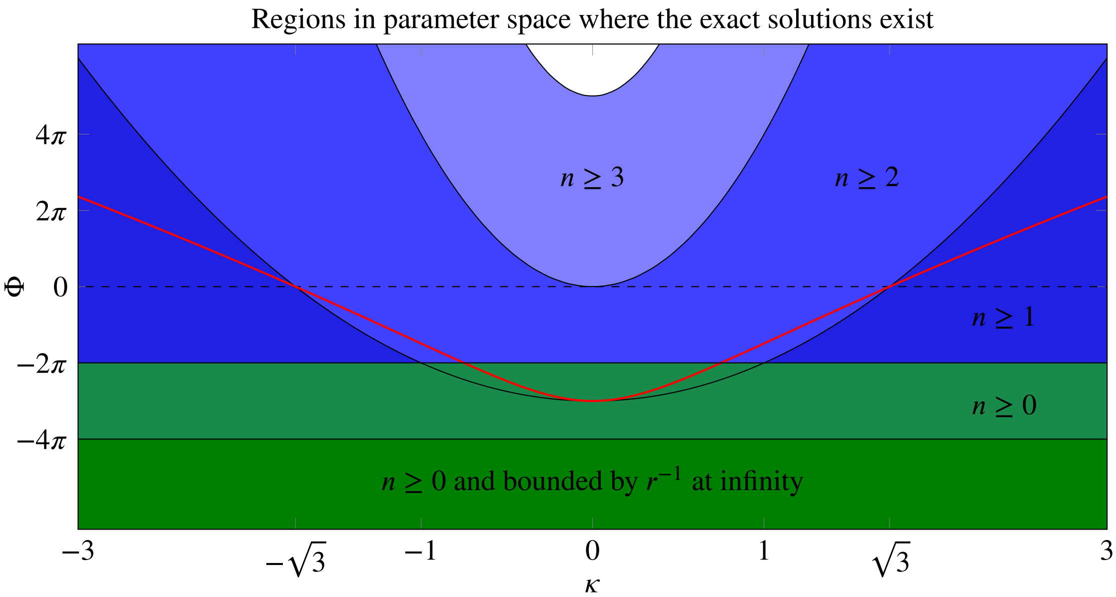

Figure 1: Regions in the (κ,Φ)-plane where the exact solutions $\boldsymbol{u}_{\Phi,n,\kappa}$ and $\boldsymbol{u}_{\Phi,\mu,A}$ exist. For $n \geq 1$, the exact solutions $\boldsymbol{u}_{\Phi,n,\kappa}$ exist in the region below the parabolas filled in blue. For $n = 0$, the solution $\boldsymbol{u}_{\Phi,\mu,A}$ exists for $\Phi < -2\pi$ and decays like r^{-1} if $\Phi \leq -4\pi$; these regions shown colored in green. The red curve represents the critical line above which the linearization of the Navier–Stokes equations around the harmonic function (3) with $\mu = -4\kappa$ decays to infinity faster than r^{-1}.

A. Monotonicity of H

In this appendix, we prove that the function $H : [0,1) \to \left(-\infty, \frac{\pi^2}{4}\right]$ defined by (19) is bijective. Since $K : [0,1) \to \left[\frac{\pi}{2}, \infty\right)$ and $E : [0,1] \to \left[\frac{\pi}{2}, 1\right]$ are bijective functions (Byrd & Friedman, 1971, #111), we find that

$$H(0) = \frac{\pi^2}{4}\,, \qquad\qquad \lim_{\alpha\to 1} H(\alpha) = -\infty\,.$$

So in order to prove that H is bijective, it is sufficient to show that $H'(\alpha) < 0$ for $\alpha \in (0,1)$. Then, by Byrd & Friedman (1971, #710.00 & 710.02), we have

$$\frac{\alpha\left(1-\alpha^2\right)}{K(\alpha)^2} H'(\alpha) = 3R(\alpha)^2 - 2\left(2-\alpha^2\right) R(\alpha) + 1 - \alpha^2\,,$$

where $R(\alpha) = E(\alpha)/K(\alpha)$. Since $K(\alpha) \geq 0$, $H'(\alpha) < 0$ if and only if

$$\frac{1}{3}\left(2-\alpha^2-\sqrt{2-\alpha^2\left(1-\alpha^2\right)}\right) < R(\alpha) < \frac{1}{3}\left(2-\alpha^2+\sqrt{2-\alpha^2\left(1-\alpha^2\right)}\right). \tag{24}$$

We remark that the bound on R given in Anderson *et al.* (1990, Thm. 2.2, (4)) is not sufficient to prove (24), because the upper bound needed here has to be tight near $\alpha = 0$. In the rest of this appendix, we prove that

$$1-\alpha^2 < R(\alpha) < 1-\frac{\alpha^2}{2}\,, \tag{25}$$

which implies (24), by using the following inequality:

$$\sqrt{2-\alpha^2\left(1-\alpha^2\right)} \leq \max\left(1-\frac{\alpha^2}{2}, 2\alpha^2-1\right).$$

By using Byrd & Friedman (1971, #710.04), we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(E(\alpha)-\left(1-\alpha^2\right)K(\alpha)\right) = \alpha K(\alpha) > 0\,,$$

so, since in addition $E(0) = K(0)$, we have $E(\alpha) - \left(1-\alpha^2\right)K(\alpha) > 0$, which corresponds to the lower bound of (25). Again by using Byrd & Friedman (1971, #710.00 & 710.02),

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(E(\alpha)-\left(1-\frac{\alpha^2}{2}\right)K(\alpha)\right) = \frac{\alpha}{2\left(1-\alpha^2\right)}\left[\left(1-\alpha^2\right)K(\alpha)-E(\alpha)\right] < 0\,,$$

so in the same way, we find that $E(\alpha) - \left(1-\frac{\alpha^2}{2}\right)K(\alpha) < 0$, which is the upper bound of (25).

Acknowledgment

The authors would like to thank one of the referees for his comments, especially concerning the interpretation in terms of one-dimensional subgroups. This work was partially supported by Swiss National Science Foundation grant 140305.

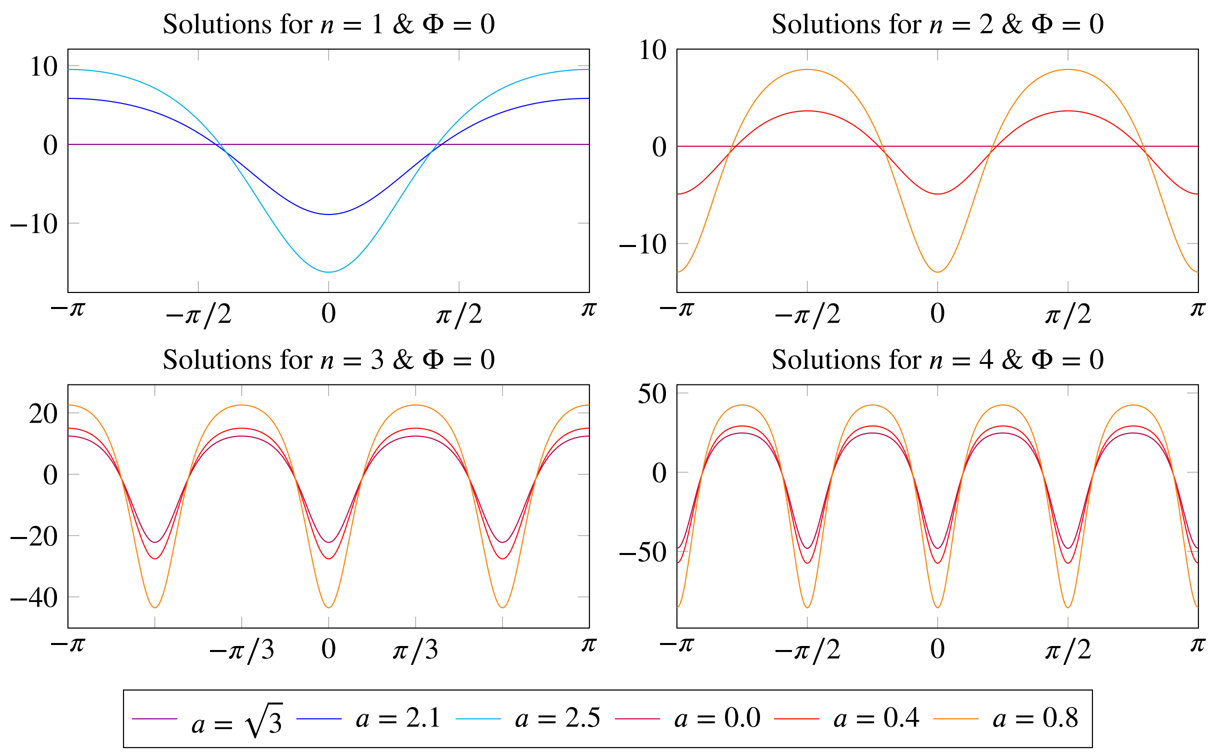

Figure 2: Periodic solutions φ of the differential equation (12) for $\Phi = 0$ and $n \in \{1, 2, 3, 4\}$. For $n = 1$, the solution exists for $|\kappa| > \sqrt{3}$, which is why in this case the values of κ start at $\sqrt{3}$. As shown the solutions are $\frac{2\pi}{n}$-periodic.

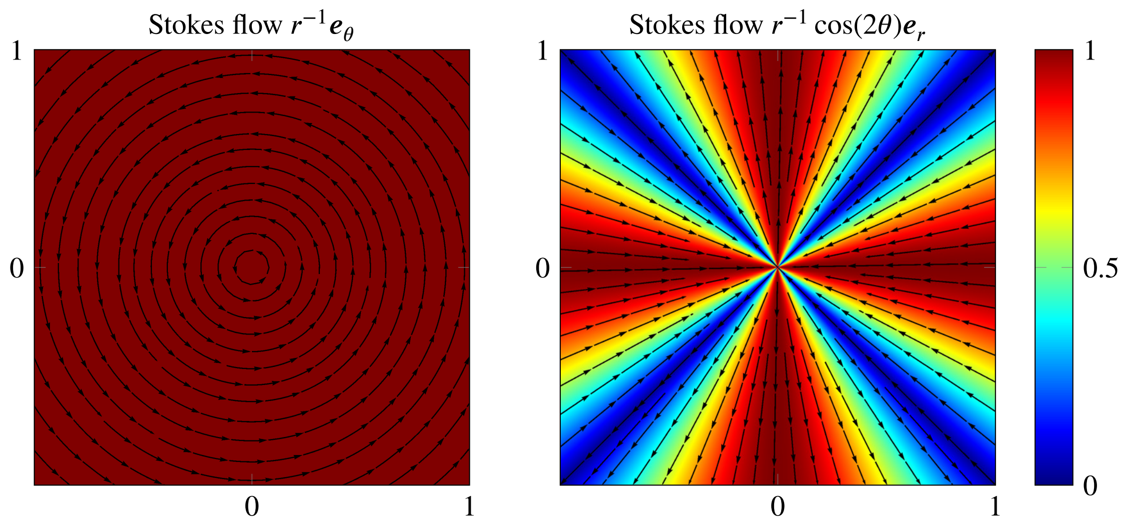

Figure 3: Representation of the velocity vector field produced by the two solutions of the Stokes equations decaying like r^{-1} . The first is generated by the torque $\int_{\mathbb{R}^2} (\boldsymbol{x} \wedge \boldsymbol{f})$ and the second one by $\int_{\mathbb{R}^2} (x_1 f_1 - x_2 f_2)$.

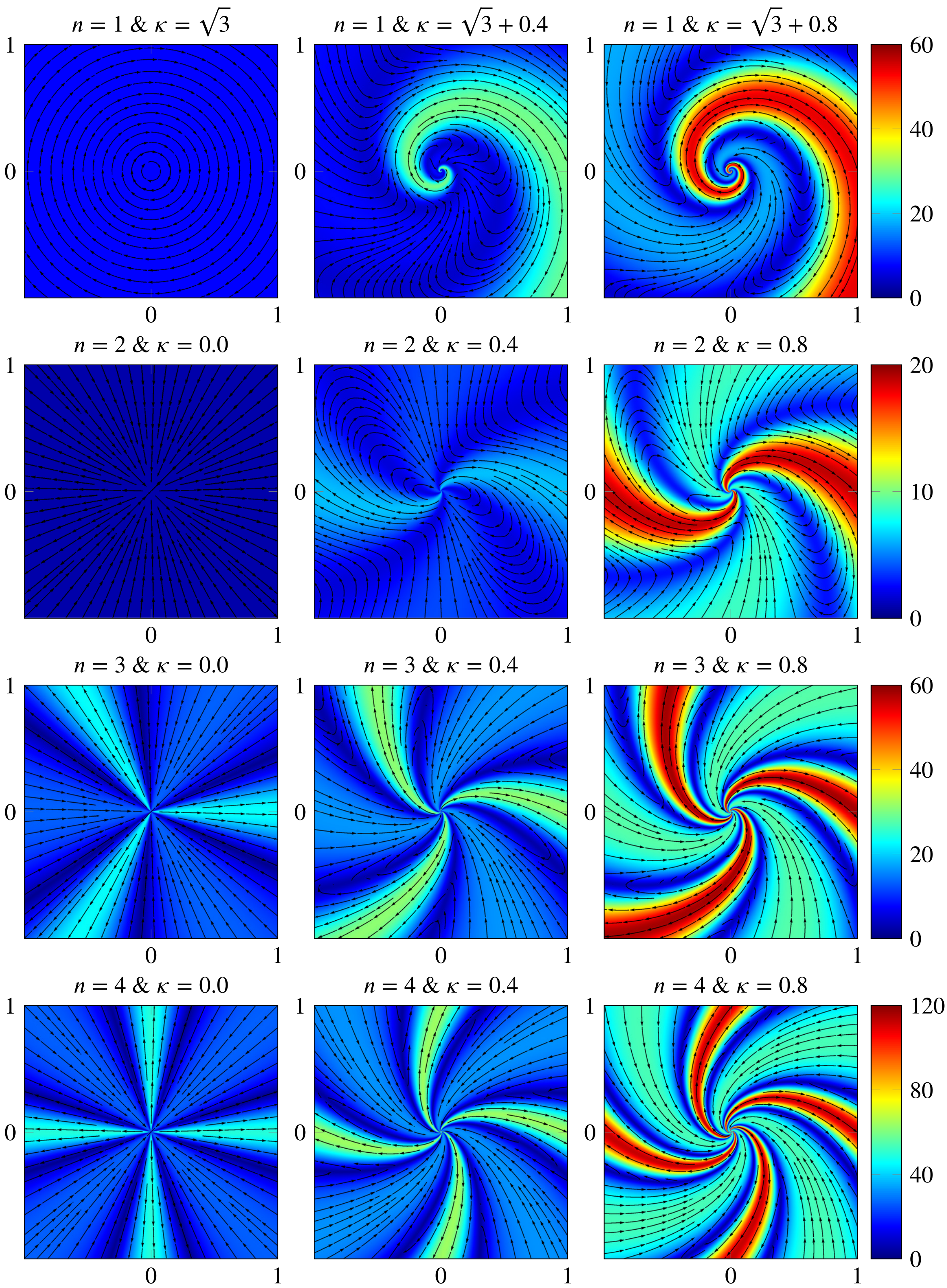

Figure 4: Representation of the velocity vector field (7) in case of zero flux, $\Phi = 0$, for different values of n and κ. The black lines represent the streamlines and the color the strength of the field $r\,|\boldsymbol{u}|$. By increasing the value of $|\kappa|$, the n branches where the velocity is high rotate more rapidly.

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