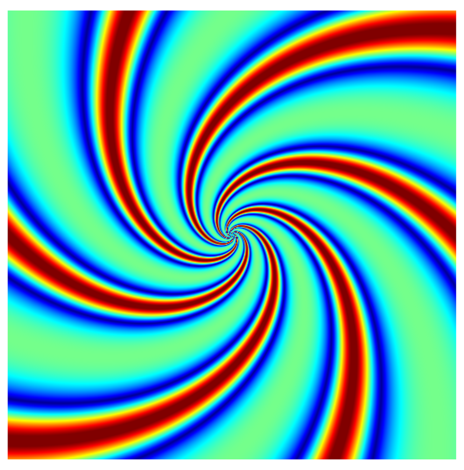

New spiral solutions for the Navier-Stokes equations

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Navier-Stokes system

The stationary Navier-Stokes equations in a two-dimensional unbounded domain Ω are

$$\Delta \boldsymbol{u} - \nabla p - (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \boldsymbol{f} \,,$$

$$\nabla \cdot \boldsymbol{u} = 0 \,, \qquad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}^* \,.$$

More precisely, the domain Ω is taken as the complement of a compact or empty set. In the case where Ω is an exterior domain, the flux Φ is an important parameter,

$$\Phi = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \,.$$

Open problem

The existence of solutions to the Navier-Stokes equations satisfying

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} = 0$$

is only known when the domain, the boundary condition $\boldsymbol{u}^*$ and the source-term $\boldsymbol{f}$ satisfy symmetries. The general case is still an open question mostly due to the fact the asymptotic behavior of weak-solutions is not known.

Symmetries

The Navier-Stokes equations are invariant under three continuous symmetries:

- Translation symmetry

$$\boldsymbol{u}(\boldsymbol{x}) \mapsto \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{x}_0) \,, \qquad \boldsymbol{x}_0 \in \mathbb{R}^2 \,.$$

- Rotation symmetry, $\mathbf{R}_\vartheta \in \mathrm{SO}(2)$

$$\boldsymbol{u}(\boldsymbol{x}) \mapsto \mathbf{R}_\vartheta^{-1} \boldsymbol{u}(\mathbf{R}_\vartheta \boldsymbol{x}) \,, \qquad \vartheta \in \mathbb{R} \,.$$

- Scaling symmetry

$$\boldsymbol{u}(\boldsymbol{x}) \mapsto \mathrm{e}^{\lambda} \boldsymbol{u}(\mathrm{e}^{\lambda} \boldsymbol{x}) \,, \qquad \lambda \in \mathbb{R} \,.$$

Scale-invariance

The only scale-invariant solutions,

$$\boldsymbol{u}(\boldsymbol{x}) = \mathrm{e}^{\lambda} \boldsymbol{u}(\mathrm{e}^{\lambda} \boldsymbol{x}) \,, \qquad \forall \lambda \in \mathbb{R} \,,$$

having a flux $\Phi \in \mathbb{R}$ are the harmonic functions

$$\boldsymbol{u}_{\mathrm{harm}} = \frac{\Phi}{2\pi r} \boldsymbol{e}_r + \frac{\mu}{r} \boldsymbol{e}_\theta \,,$$

for $\mu \in \mathbb{R}$ and the Hamel solutions

$$\boldsymbol{u}_{\mathrm{hamel}} = \frac{-1}{r} \varphi(\theta_0 + \theta) \boldsymbol{e}_r \,,$$

for $n \in \mathbb{N}^*$ and $4 + \frac{\Phi}{\pi} \leq n^2$, where φ is a $\frac{2\pi}{n}$-periodic function.

Asymptotic behavior

- In three-dimensions, Šverák proves that the solution of the Navier-Stokes equations behaves at infinity like a scale-invariant solution, *i.e.* a Landau solution.
- In two-dimensions, he also proves that, even if the source term has zero mean, one cannot prove that the asymptote is a scale-invariant solution, *i.e.* a Hamel solution.

New symmetry

The idea is to combine the rotational and the scaling symmetries:

$$G = \mathbb{R} \times \mathrm{SO}(2) \simeq \mathbb{R} \times S^1 \,.$$

One-dimensional subgroups of G:

$$G_\kappa = \{(\lambda, \vartheta) \in G \,:\, \kappa\lambda + \vartheta = 0\} \,,$$

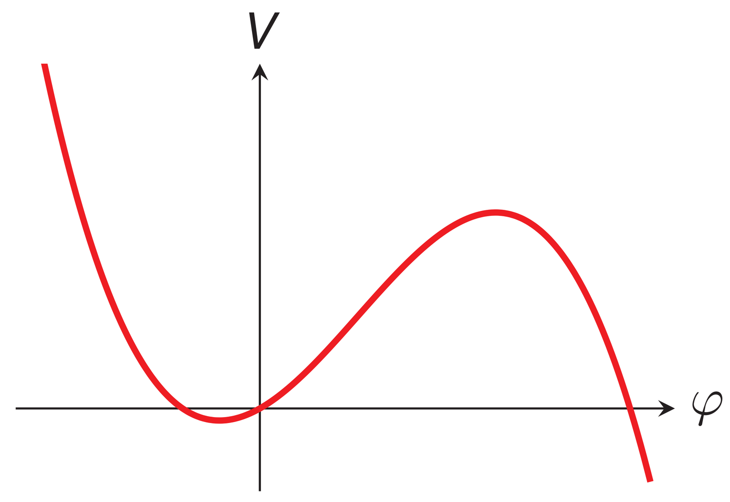
where $\kappa \in \mathbb{R}$ or $\kappa = \infty$.
A solution is invariant under G_κ if

$$\boldsymbol{u}(\boldsymbol{x}) = \mathrm{e}^{\lambda} \mathbf{R}_{\kappa\lambda} \boldsymbol{u}(\mathrm{e}^{\lambda} \mathbf{R}_{-\kappa\lambda} \boldsymbol{x}) \,, \qquad \lambda \in \mathbb{R} \,.$$

This generalizes the scale-invariant solutions ($\kappa = 0$) and the rotation-invariant solutions ($\kappa = \infty$).

Potential

The ordinary differential equation assimilates to the Newton equation of a damped particle in a potential:



In order obtain periodic solutions, the damping term $\mu + 4\kappa$ has to vanish.

Main theorem on spiral solutions

For all $\Phi, \kappa \in \mathbb{R}$, and $n \in \mathbb{N}^*$ satisfying $4 + \frac{\Phi}{\pi} \leq n^2 \left(1 + \kappa^2\right)$ there exists a $\frac{2\pi}{n}$-periodic function φ depending on n, Φ, and κ such that for any $\theta_0 \in \mathbb{R}$,

$$\boldsymbol{u}_{\mathrm{new}} = \frac{1}{r} \Big[-\varphi(\theta + \kappa \log r + \theta_0) \boldsymbol{e}_r + \kappa \left(\varphi(\theta + \kappa \log r + \theta_0) - 4 \right) \boldsymbol{e}_\theta \Big]$$

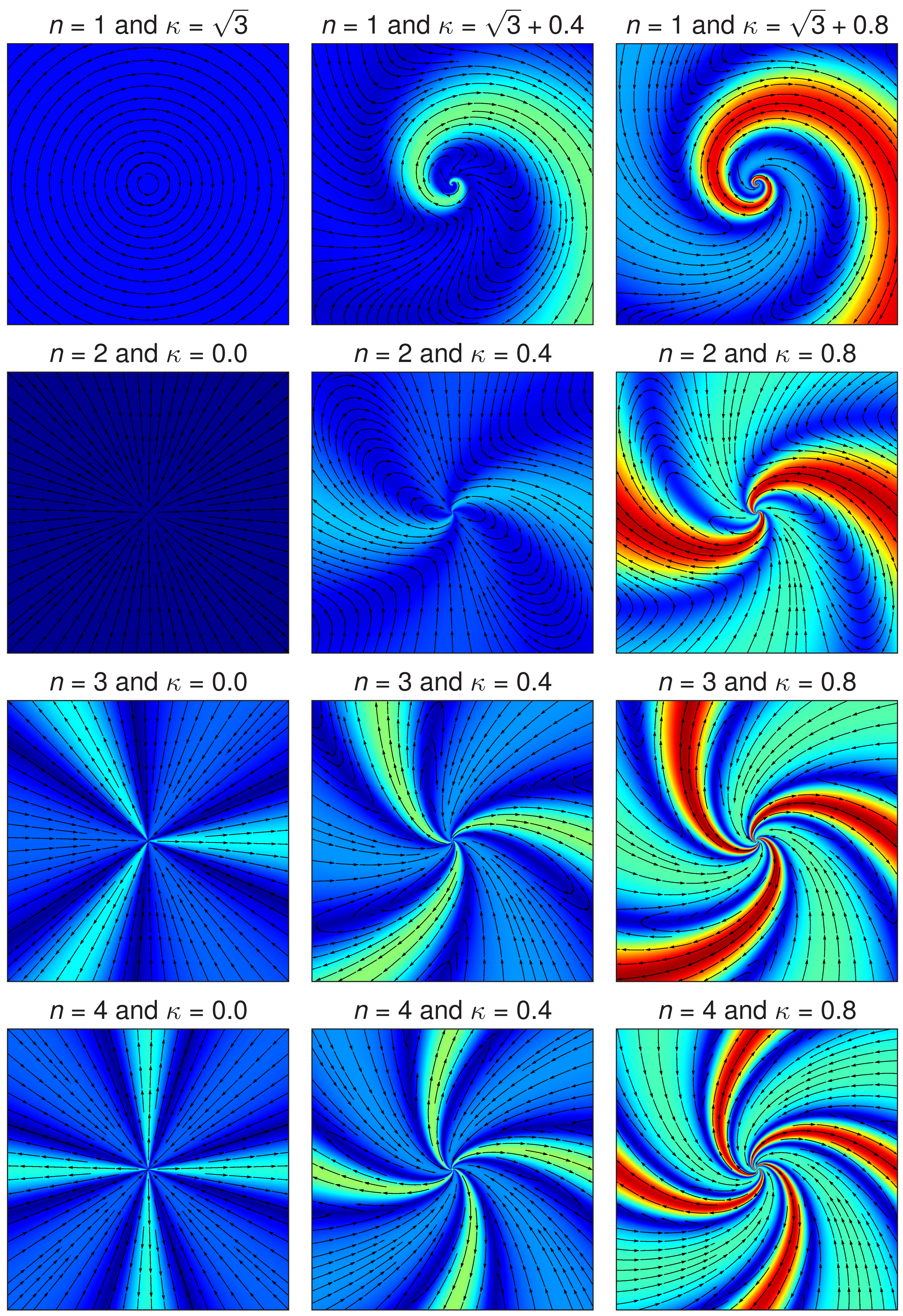
is a solution of the Navier-Stokes equations having flux Φ. Moreover, $\boldsymbol{u}_{\mathrm{new}}$ and to $\boldsymbol{u}_{\mathrm{harm}}$ are the only invariant solutions under the subgroup G_κ.

Ordinary differential equation

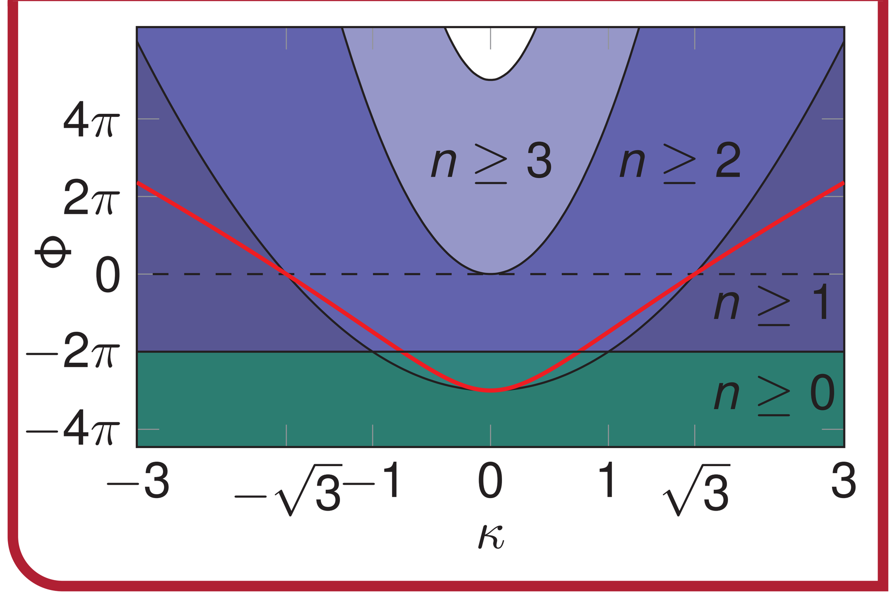
The function φ in Theorem 1 has to be a 2π-periodic solution of

$$\left(1 + \kappa^2\right) \varphi''(z) - (\mu + 4\kappa)\, \varphi'(z) + 4\varphi(z) = \varphi(z)^2 - C \,.$$

New solutions exhibit spiral behavior in r^{-1}



Regions of existence



Small solutions

The only scale-invariant solutions up to a rotation that have zero flux and have $\|r\boldsymbol{u}\|_\infty$ small are the harmonic solution $\boldsymbol{u}_{\mathrm{harm}}$ and the new solution $\boldsymbol{u}_{\mathrm{new}}$ for $n = 2$.

Asymptotic behavior

If $\boldsymbol{f}$ has zero mean, the Stokes approximation

$$\Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} \,, \qquad \nabla \cdot \boldsymbol{u} = 0 \,.$$

has the asymptotic expansion

$$\boldsymbol{u} = \frac{1}{r} \left[A \cos(2\theta) \boldsymbol{e}_r + M \boldsymbol{e}_\theta \right] + O(r^{-2}) \,,$$

where A and M are compatibility conditions,

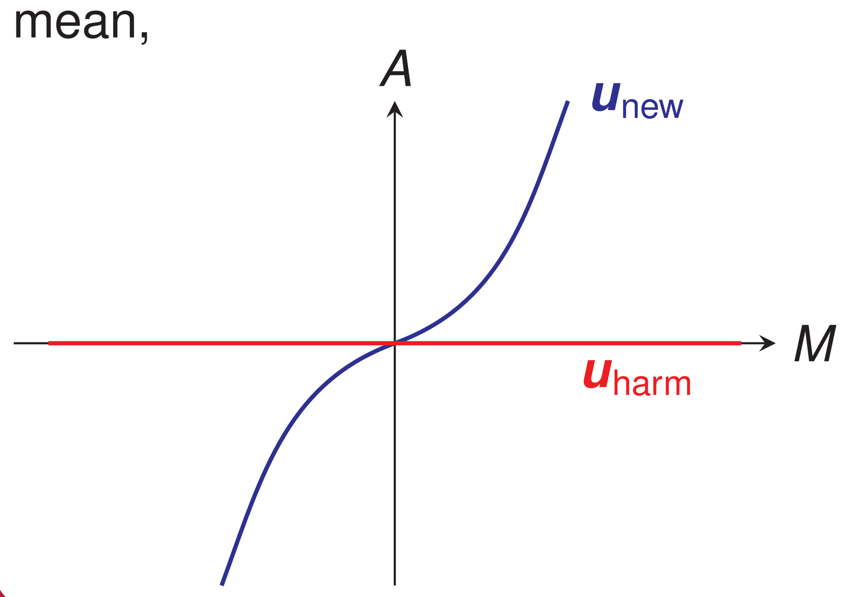
$$A = \int_{\mathbb{R}^2} (x_1 f_1 - x_2 f_2) \,,$$

and

$$M = \int_{\mathbb{R}^2} (\boldsymbol{x} \wedge \boldsymbol{f}) \,.$$

Branching

The new solutions are not sufficient to describe the general asymptotic behavior even in the case $\boldsymbol{f}$ has zero mean,



Conclusions

1. New method for finding exact solutions by using symmetries.
2. New solutions decaying like r^{-1} with spiral streamlines.
3. Lack of a parameter to describe the asymptotic behavior in the case where $\boldsymbol{f}$ has zero mean.

References

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