ASYMPTOTIC BEHAVIOR OF A VISCOUS FLOW PAST A BODY

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VISCOUS FLOW PAST A BODY

- Exterior domain:

$$\Omega = \mathbb{R}^2 \setminus \overline{B} \quad \text{with} \quad B \quad \text{bounded}$$

- Stationary Navier–Stokes equations:

$$\Delta \boldsymbol{u} - \nabla p - \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{f} \qquad \nabla \cdot \boldsymbol{u} = 0$$

- Boundary conditions:

$$\boldsymbol{u}|_{\partial B} = \boldsymbol{0} \quad \text{and} \quad \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{u} = \boldsymbol{u}_\infty$$

Main question
What is the asymptotic behavior of the solutions at large distances for $\boldsymbol{u}_\infty \neq \boldsymbol{0}$? WLOG assume that $\boldsymbol{u}_\infty = 2\boldsymbol{e}_1$.

EXISTENCE OF SOLUTIONS

Weak solution (Leray, 1933)

Existence of a weak solution $\boldsymbol{u} \in \dot{H}^1_0(\Omega)$ even for large data. However, the convergence of $\boldsymbol{u}$ to $\boldsymbol{u}_\infty$ at large distances is open.

Physically reasonable solution (Finn, 1965)

Existence of a physically reasonable solution

$$\boldsymbol{u} = \boldsymbol{u}_\infty + O(|\boldsymbol{x}|^{-1/4-\varepsilon})$$

with $\varepsilon > 0$ for small data. The existence of physically reasonable solutions is open for large data.

ASYMPTOTE OF THE VELOCITY

Theorem (Babenko, 1970)

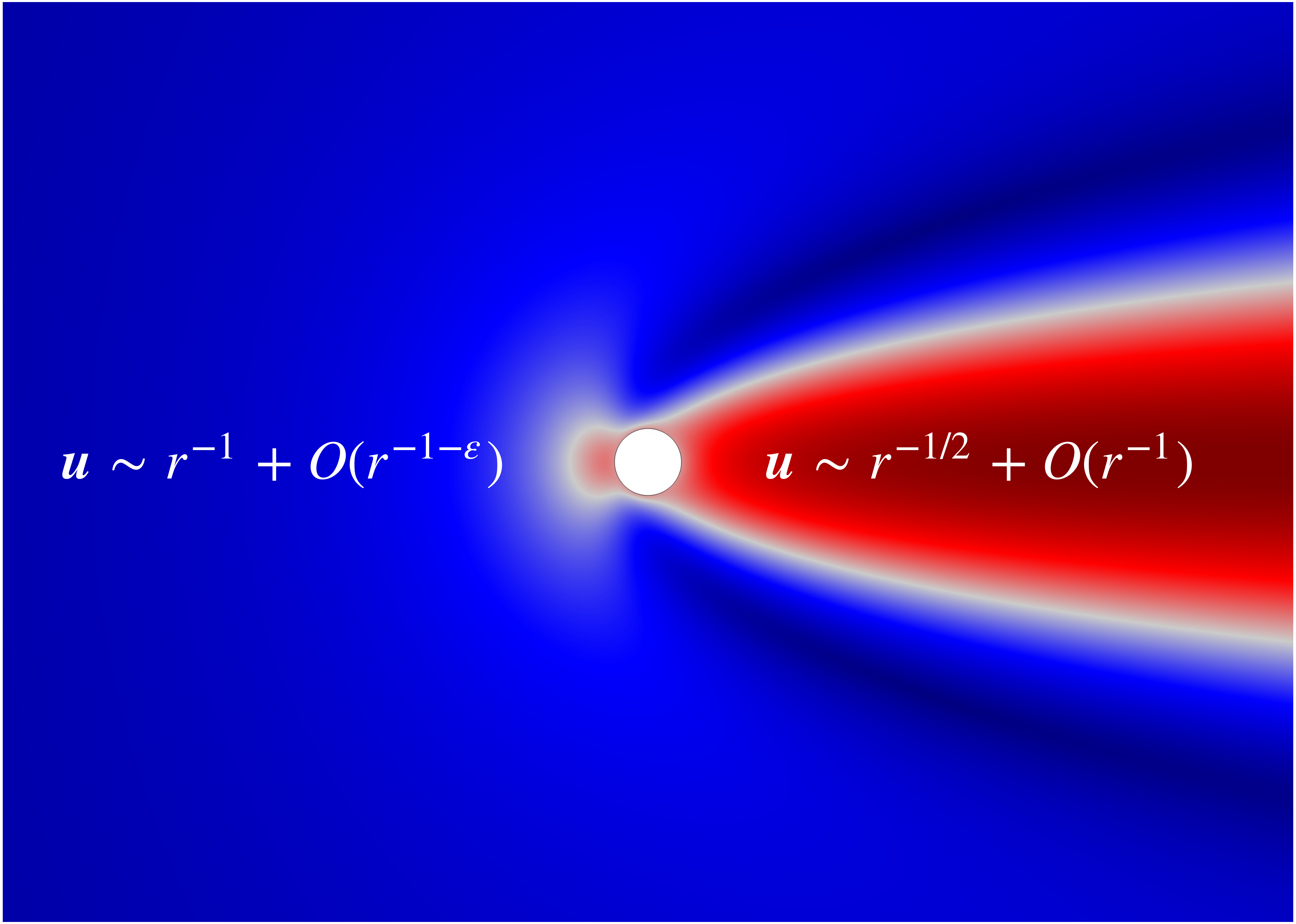
If $\boldsymbol{u}$ is a physically reasonable solution, then

$$\boldsymbol{u} = \boldsymbol{u}_\infty - \sqrt{2\pi} F_1 \frac{\mathrm{e}^{-r(1-\cos\theta)}}{r^{1/2}} + 2F_1 \frac{\boldsymbol{e}_r}{r} - 2F_2 \frac{\boldsymbol{e}_\theta}{r} + O\left(\left(\frac{|\theta| \log r}{r^{1/2}} + \frac{1}{r} \right) \mathrm{e}^{-r(1-\cos\theta)} + \frac{1}{r^{1+\varepsilon}} \right)$$

where $\boldsymbol{F} \in \mathbb{R}^2$ is the net force acting on the body.

Remark: linear asymptote

This asymptotic behavior is given by the Oseen system, which is the linearization around $\boldsymbol{u}_\infty$.


$\boldsymbol{u} \sim r^{-1} + O(r^{-1-\varepsilon})$
$\boldsymbol{u} \sim r^{-1/2} + O(r^{-1})$

ASYMPTOTE OF THE VORTICITY

Theorem (Babenko, 1970)

If $\boldsymbol{u}$ is a physically reasonable solution, then

$$\omega = \nabla \wedge \boldsymbol{u}(\boldsymbol{x}) = -\sqrt{8\pi} F_1 \sin\theta \frac{\mathrm{e}^{-r(1-\cos\theta)}}{r^{1/2}} + O\left(\frac{1}{r^{3/2}} \mathrm{e}^{-\mu r(1-\cos\theta)}\right)$$

for any $\mu \in (0, 1)$.

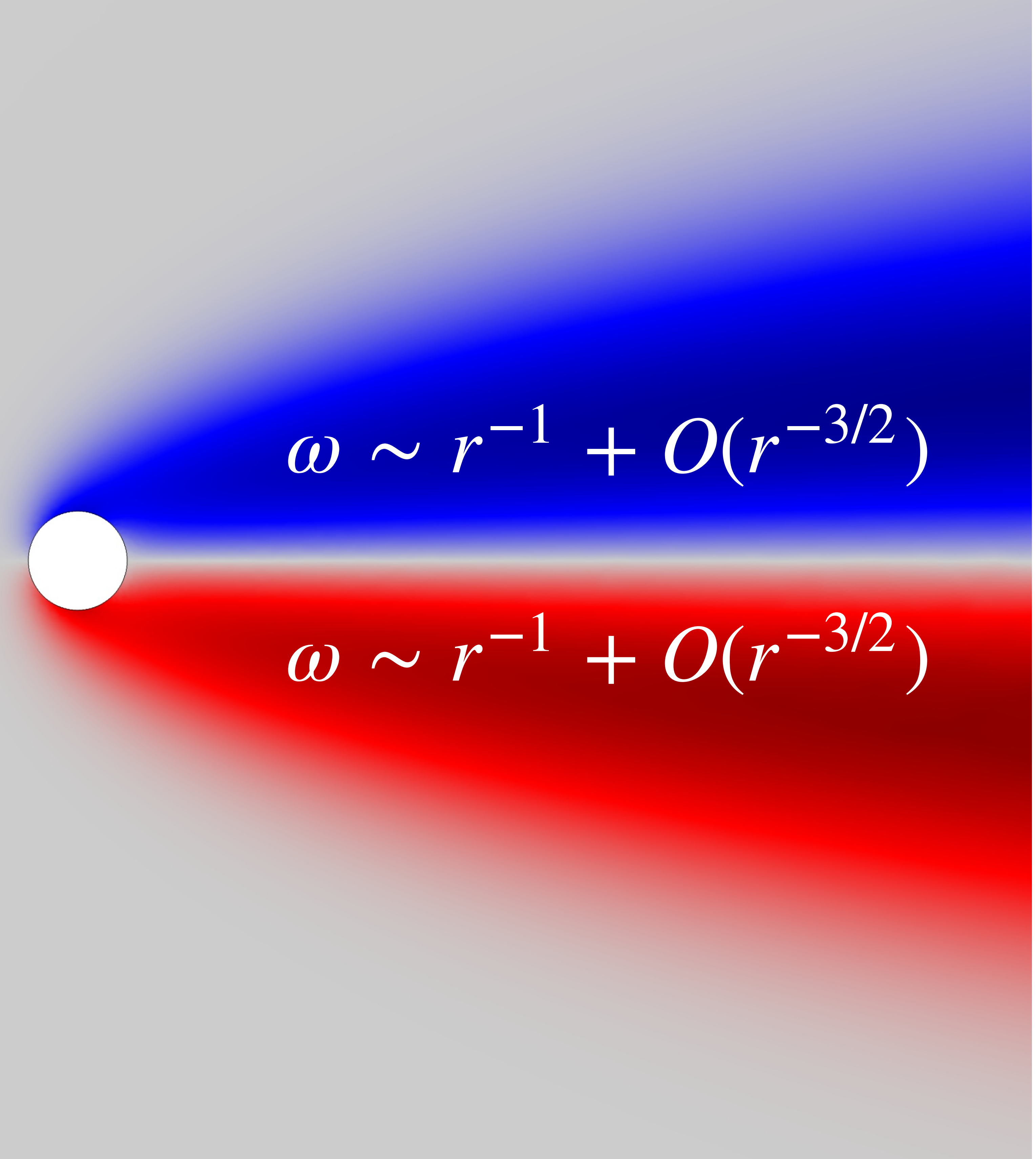
Remark: linear asymptote

This asymptotic behavior is obtained by a complicated bootstrap from the linearization around $\boldsymbol{u}_\infty$.

Problem

No main asymptotic term outside the wake, only remainder.

$$\omega \sim O\left(r^{-3/2}\mathrm{e}^{-\mu r(1-\cos\theta)}\right)$$



Question

How does the vorticity behave outside the wake region?

OPTIMAL ASYMPTOTE OF ω

Theorem (Guillod & Wittwer, 2016)

If $\boldsymbol{u}$ is a physically reasonable solution, then

$$\omega = r^{F_1(1-\cos\theta)-F_2\sin\theta}\left(\mu(\theta)+O(r^{-\varepsilon})\right)\frac{\mathrm{e}^{-r(1-\cos\theta)}}{r^{1/2}}$$

where $\mu \in C^{2\varepsilon}(S^1)$ is a 2π-periodic function depending on the data.

Remark: optimality

This asymptotic expansion is now optimal: the remainder decays faster than the asymptote in any region.

Remark: nonlinear asymptote

This asymptotic behavior is related to the nonlinearity and is not given by the linearization around $\boldsymbol{u}_\infty$.

Remark: nonuniversal asymptote

The asymptotic behavior of the vorticity is not universal, the polynomial power of decay in front of the exponential factor depends on the data through the net force $\boldsymbol{F}$.

Remark: only in two dimensions

The nonlinear and nonuniversal nature of the asymptote of the vorticity is specific to the two-dimensional case. In three dimensions, everything is easier and known.

DIFFICULTIES

- The asymptote having a nonlinear feature, we cannot use a bootstrap argument from the linearization around $\boldsymbol{u}_\infty$.
- The exponential factor $\mathrm{e}^{-r(1-\cos\theta)}$ is critical in the vorticity equation.
- The theorem holds for any physically reasonable solution, even the large ones.

IDEAS OF THE PROOF

- Use the vorticity equation:

$$\Delta\omega - \boldsymbol{u}\cdot\nabla\omega = \nabla\wedge\boldsymbol{f} \qquad \omega = \nabla\cdot\boldsymbol{u}$$

- View the vorticity equation as a linear equation in a large enough ball B_R with $\boldsymbol{u}$ and $\omega|_{\partial B_R}$ fixed:

$$\Delta w - \boldsymbol{u}\cdot\nabla w = \nabla\wedge\boldsymbol{f} \qquad w|_{\partial B_R} = \omega|_{\partial B_R}$$

- By a fixed point argument (smallness coming from large R), prove the existence of a solution w satisfying:

$$|w(\boldsymbol{x})| \le C r^{F_1(1-\cos\theta)-F_2\sin\theta}\,\frac{\mathrm{e}^{-r(1-\cos\theta)}}{r^{1/2}}$$

- Prove that w has the asymptotic behavior claimed for ω.

- Prove that the linear equation satisfied by w has a unique solution, hence $\omega = w$.

ANALYSIS OF THE EQUATION FOR w

- From the result of Babenko:
$$\boldsymbol{u} = \boldsymbol{u}_\infty + \boldsymbol{u}_h + O\left(\frac{1}{r^{1/2}}\mathrm{e}^{-r(1-\cos\theta)} + \frac{1}{r^{1+\varepsilon}}\right)$$
where
$$\boldsymbol{u}_h = 2F_1\frac{\boldsymbol{e}_r}{r} - 2F_2\frac{\boldsymbol{e}_\theta}{r} = 2\nabla\left(F_1\log r - F_2\theta\right)$$
- Change of variables:
$$w(r,\theta) = r^{A_1(1-\cos\theta)-A_2\sin\theta}\mathrm{e}^{r\cos\theta}b(r,\theta)$$
- Transformed equation:
$$\Delta b - b = \boldsymbol{v}\cdot(\nabla b + b\boldsymbol{e}_r) + hb + R$$
where $\boldsymbol{v}$ and h make the right hand side subdominant.

- The right hand side is now subcritical: everything is governed by the linear operator $\Delta - 1$.

- Existence of a solution such that

$$|b| \leq \frac{C}{r^{1/2}} e^{-r}$$

- Lengthly calculations to prove that:

$$b = \frac{1}{r^{1/2}} \left(\mu(\theta) + O(r^{-\varepsilon}) \right) e^{-r}$$

- Uniqueness: for $\boldsymbol{u} \in L^\infty(\Omega)$, if $w \in \dot{H}^1_0(\Omega)$ is a solution of

$$\Delta w - \boldsymbol{u} \cdot \nabla w = 0$$

and $w \in L^4(\Omega)$, then $w = 0$.

INTUITION

Harmonic transport

Let $\boldsymbol{u}_h$ be an an harmonic divergence-free vector field

$$\boldsymbol{u}_h = 2\nabla^{\perp}\psi = 2\nabla\phi$$

Formally, the equation

$$\Delta w - \boldsymbol{u}_h \cdot \nabla w = g$$

is transformed into

$$\Delta b - b = h$$

by the change of variables

$$w(\boldsymbol{x}) = b(\phi(\boldsymbol{x}), \psi(\boldsymbol{x}))\, \mathrm{e}^{-\phi(\boldsymbol{x})}$$

$$g(\boldsymbol{x}) = |\nabla\phi(\boldsymbol{x})|^2 h(\boldsymbol{x})\, \mathrm{e}^{-\phi(\boldsymbol{x})}$$

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