Méthodes numériques pour les EDP instationnaires

TD 1: jeudi 09.09.2021
Transport equation with constant coefficients

Solution

For a given $a \in \mathbb{R}$, we consider the following linear transport equation in one dimension :

$$\begin{cases} \partial_t \bar{u} + a\, \partial_x \bar{u} = 0\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ \bar{u}(x,0) = u_0(x)\,, & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u_0 \in L^\infty(\mathbb{R})$. Without loss of generality, we assume that $a > 0$. We refer to the chapter 2, subsection 2.2.1, for the continuous framework of this equation. Here we focus on finding u a discrete approximation of $\bar{u}$ thanks to discrete schemes. As in chapter 3, we introduce a discretization of the domain using a regular mesh : $(x_j, t_n) = (j\Delta x, n\Delta t)$, $\forall j \in \mathbb{Z}$, $\forall n \in \mathbb{N}$, where Δx, respectively Δt, denotes the space step, respectively the time step. We also denote u_j^n the approximation of $\bar{u}(x_j, t_n)$.

Definition: A scheme is L^∞ stable if we can prove the estimate

$$\sup_j \left|u_j^{n+1}\right| \le \sup_j \left|u_j^n\right|.$$

Definition: A scheme is L^2 stable if we can prove the estimate

$$\sum_j \left|u_j^{n+1}\right|^2 \le \sum_j \left|u_j^n\right|^2.$$

1 Lax-Wendroff scheme

We first focus on the *Lax-Wendroff* scheme :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2 \Delta t}{2}\,\frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} = 0\,. \tag{2}$$

Q1: Truncation error
The exact solution $\bar{u}$ of (1) is generally not a solution of the scheme (2). The truncation error estimates the difference. Let us assume that the solution of (1) is such that $\bar{u} \in C^3(\mathbb{R} \times \mathbb{R}^+)$.

1. Prove that, for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $\partial_{tt}\bar{u} = a^2\, \partial_{xx}\bar{u}$.

Taking respectivly the derivatives with respect to t and x of the equation leads to

$$\partial_{tt}\bar{u} + a\, \partial_{tx}\bar{u} = 0\,,$$
$$\partial_{tx}\bar{u} + a\, \partial_{xx}\bar{u} = 0\,,$$

so eliminating $\partial_{tx}\bar{u}$, we obtain

$$\partial_{tt}\bar{u} = a^2\, \partial_{xx}\bar{u}\,.$$

2. Compute the Taylor expansions ("développements limités avec reste de Taylor-Lagrange") at a convenient order of $\bar{u}(x_j, t_{n+1})$, $\bar{u}(x_{j+1}, t_n)$, and $\bar{u}(x_{j-1}, t_n)$ at the point (x_j, t_n).

We denote by $\bar{u}_j^n$ the evaluation of $\bar{u}$ at x_j, t_n,

$$\bar{u}_j^n = \bar{u}(x_j, t_n)\,.$$

Using Taylor expansions, one has

$$\bar{u}_j^{n+1} = \bar{u}(x_j, t_{n+1}) = \bar{u}(x_j, t_n + \Delta t) = \bar{u}(x_j, t_n) + \Delta t\, \partial_t \bar{u}(x_j, t_n) + \frac{\Delta t^2}{2}\partial_{tt}\bar{u}(x_j, t_n) + O(\Delta t^3)\,,$$
$$\bar{u}_{j+1}^n = \bar{u}(x_j + \Delta x, t_n) = \bar{u}(x_j, t_n) + \Delta x\, \partial_x \bar{u}(x_j, t_n) + \frac{\Delta x^2}{2}\partial_{xx}\bar{u}(x_j, t_n) + O(\Delta x^3)\,,$$
$$\bar{u}_{j-1}^n = \bar{u}(x_j - \Delta x, t_n) = \bar{u}(x_j, t_n) - \Delta x\, \partial_x \bar{u}(x_j, t_n) + \frac{\Delta x^2}{2}\partial_{xx}\bar{u}(x_j, t_n) + O(\Delta x^3)\,.$$

At this stage, it is no so clear up to which order one has to perform the expansions.

3. Assuming that enough partial derivatives of $\bar{u}$ are bounded in L^∞ norm by some constant $C \in \mathbb{R}_*^+$, prove that the absolute value of the truncation error of the Lax-Wendroff scheme is second order both in time and space.

Using the previous expansions, one has

$$\frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} = \partial_t \bar{u}(x_j, t_n) + \frac{\Delta t}{2}\partial_{tt}\bar{u}(x_j, t_n) + O(\Delta t^2)\,,$$
$$\frac{\bar{u}_{j+1}^n - \bar{u}_{j-1}^n}{2\Delta x} = \partial_x \bar{u}(x_j, t_n) + O(\Delta x^2)\,,$$
$$\frac{2\bar{u}_j^n - \bar{u}_{j-1}^n - \bar{u}_{j+1}^n}{\Delta x^2} = -\partial_{xx}\bar{u}(x_j, t_n) + O(\Delta x)\,.$$

so, combining the three terms leads to

$$\begin{aligned} T_j^n &= \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} + a\frac{\bar{u}_{j+1}^n - \bar{u}_{j-1}^n}{2\Delta x} + \frac{a^2\Delta t}{2}\frac{2\bar{u}_j^n - \bar{u}_{j-1}^n - \bar{u}_{j+1}^n}{\Delta x^2} \\ &= \partial_t \bar{u}(x_j, t_n) + \frac{\Delta t}{2}\partial_{tt}\bar{u}(x_j, t_n) + a\partial_x \bar{u}(x_j, t_n) - \frac{\Delta t}{2}a^2\partial_{xx}\bar{u}(x_j, t_n) + O(\Delta t^2) + O(\Delta x^2) + O(\Delta x \Delta t) \\ &= \underbrace{\partial_t \bar{u}(x_j, t_n) + a\partial_x \bar{u}(x_j, t_n)}_{=0} + \frac{\Delta t}{2}\Big[\underbrace{\partial_{tt}\bar{u}(x_j, t_n) - a^2\partial_{xx}\bar{u}(x_j, t_n)}_{=0}\Big] + O(\Delta t^2) + O(\Delta x^2) + O(\Delta x \Delta t) \\ &= O(\Delta t^2 + \Delta x^2)\,. \end{aligned}$$

In the last step, we used the equation, the result of the previous point, and the fact that:

$$|\Delta x \Delta t| \le \frac{1}{2}\left(\Delta x^2 + \Delta t^2\right)\,.$$

We note that the $O(\Delta t^2 + \Delta x^2)$ depends on a and on the L^∞-norms of $\partial_{xxx}\bar{u}$ and $\partial_{ttt}\bar{u}$. More precisely, this means, that there exists a constant $C > 0$ depending on a and these L^∞-norms, such that

$$|T_j^n| \le C(\Delta t^2 + \Delta x^2)\,.$$

Q2: L^∞ stability

1. Show that, for any non-negative values α, β, γ such that $\alpha + \beta + \gamma = 1$, then

$$\forall x, y, z \in \mathbb{R}, \min(x, y, z) \le \alpha x + \beta y + \gamma z \le \max(x, y, z).$$

Let $x, y, z \in \mathbb{R}$ and

$$M = \max(x, y, z)\,, \qquad m = \min(x, y, z)\,.$$

Since $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma = 1$, we have

$$\alpha x + \beta y + \gamma z \le \alpha M + \beta M + \gamma M \le M\,,$$
$$\alpha x + \beta y + \gamma z \ge \alpha m + \beta m + \gamma m \ge m\,.$$

Therefore, the inequality is proven. In particular, we note that

$$|\alpha x + \beta y + \gamma z| \le \max(|x|, |y|, |z|)\,.$$

2. Using (2), find α, β, γ such that $u_j^{n+1} = \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n$.

Using (2) and denoting the CFL by

$$\nu = a\frac{\Delta t}{\Delta x}\,.$$

we have

$$\begin{aligned} u_j^{n+1} &= u_j^n - a\frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) - a^2\frac{\Delta t^2}{2\Delta x^2}\left(2u_j^n - u_{j-1}^n - u_{j+1}^n\right) \\ &= u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{\nu^2}{2}\left(2u_j^n - u_{j-1}^n - u_{j+1}^n\right) \\ &= \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n\,, \end{aligned}$$

where

$$\alpha = 1 - \nu^2\,, \qquad \beta = \frac{\nu^2}{2} - \frac{\nu}{2}\,, \qquad \gamma = \frac{\nu^2}{2} + \frac{\nu}{2}\,.$$

One can check that $\alpha + \beta + \gamma = 1$.

3. Provide a necessary and sufficient condition on Δt, Δx and a ensuring the non-negativity of the coefficients α, β, γ found at the previous question. Deduce the L^∞ stability domain of the scheme.

In view of the inequality proven in the first point, if $\alpha, \beta, \gamma \ge 0$, one has

$$\sup_j |u_j^{n+1}| = \sup_j |\alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n| \le \sup_j \max\left(|u_j^n|, |u_{j+1}^n|, |u_{j-1}^n|\right) \le \sup_j |u_j^n|\,.$$

Therefore, the scheme is L^∞-stable if $\alpha, \beta, \gamma \ge 0$. By cooking up a counter-exemple, it is easy to see that if at least one of the coefficients α, β, γ is strictly negative, then the scheme is not L^∞-stable. Therefore a scheme in the form

$$u_j^{n+1} = \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n$$

is L^∞-stable if and only if $\alpha, \beta, \gamma \ge 0$. By representing graphically the coefficients α, β, γ in terms of $\nu \ge 0$, one see easily that thid condition is equivalent to $\nu = 1$.

Q3: L^2 stability

1. Show that

$$\sum_j \left|u_j^{n+1}\right|^2 = \sum_j \left|u_j^n\right|^2 - \frac{\nu^2(1-\nu^2)}{4}\sum_j \left|w_{j+1}^n - w_j^n\right|^2,$$

where $\nu = \dfrac{a\Delta t}{\Delta x}$ and $w_j^n = u_j^n - u_{j-1}^n$.

Since

$$u_j^{n+1} = \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n\,,$$

we have

$$|u_j^{n+1}|^2 = \alpha^2|u_j^n|^2 + \beta^2|u_{j+1}^n|^2 + \gamma^2|u_{j-1}^n|^2 + 2\alpha\beta u_j^n u_{j+1}^n + 2\alpha\gamma u_j^n u_{j-1}^n + 2\beta\gamma u_{j+1}^n u_{j-1}^n\,.$$

The aim is now to sum over j and express the cross terms in terms of squares. Since

$$\sum_j |u_j^n|^2 = |u_{j-1}^n|^2 = |u_{j+1}^n|^2,$$

we have

$$\sum_j u_j^n u_{j+1}^n = \frac{1}{2}\sum_j |u_j^n|^2 + \frac{1}{2}\sum_j |u_{j+1}^n|^2 - \frac{1}{2}\sum_j |u_{j+1}^n - u_j^n| = \sum_j |u_j^n|^2 - \frac{1}{2}\sum_j |w_j^n|^2\,,$$
$$\sum_j u_j^n u_{j-1}^n = \frac{1}{2}\sum_j |u_j^n|^2 + \frac{1}{2}\sum_j |u_{j-1}^n|^2 - \frac{1}{2}\sum_j |u_{j-1}^n - u_j^n| = \sum_j |u_j^n|^2 - \frac{1}{2}\sum_j |w_j^n|^2\,,$$
$$\sum_j u_{j+1}^n u_{j-1}^n = \frac{1}{2}\sum_j |u_{j+1}^n|^2 + \frac{1}{2}\sum_j |u_{j-1}^n|^2 - \frac{1}{2}\sum_j |u_{j+1}^n - u_{j-1}^n| = \sum_j |u_j^n|^2 - \frac{1}{2}\sum_j |w_{j+1}^n + w_{j-1}^n|^2\,.$$

Therefore, taking the sum over j in the expression of $|u_j^{n+1}|^2$ leads to

$$\sum_j |u_j^{n+1}|^2 = \left(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma\right)\sum_j |u_j^n|^2 - \alpha\left(\beta + \gamma\right)\sum_j |w_j^n|^2 - \beta\gamma\sum_j |w_{j+1}^n + w_{j-1}^n|^2\,.$$

Again, one has to do the same trick for the last term,

$$\sum_j |w_{j+1}^n + w_{j-1}^n|^2 = 2\sum_j |w_j^n|^2 + 2\sum_j w_{j+1}^n w_j^n = 4\sum_j |w_j^n|^2 - \sum_j |w_{j+1}^n - w_j^n|^2\,,$$

and we obtain

$$\sum_j |u_j^{n+1}|^2 = (\alpha + \beta + \gamma)^3 \sum_j |u_j^n|^2 - (\alpha\beta + \alpha\gamma + 4\beta\gamma)\sum_j |w_j^n|^2 + \beta\gamma\sum_j |w_{j+1}^n - w_j^n|^2\,.$$

Using the epressions of α, β, γ in terms of ν, we have

$$\alpha\beta + \alpha\gamma = \nu^2\left(1 - \nu^2\right) = -4\beta\gamma\,,$$

so

$$\sum_j |u_j^{n+1}|^2 = \sum_j |u_j^n|^2 - \frac{\nu^2(1-\nu^2)}{4}\sum_j |w_{j+1}^n - w_j^n|^2\,.$$

2. Deduce the condition under which the scheme is L^2 stable.

The previous equality shows that the scheme is L^2-stable if $\nu \le 1$. By constructing a counterexample, we can prove this is in fact a necessary condition.

2 Schemes overview

- *Centered explicit scheme*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0. \tag{3}$$

- *Centered implicit scheme*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0. \tag{4}$$

- *Upwind scheme*

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\dfrac{u_j^n - u_{j-1}^n}{\Delta x} = 0\,, \text{ if } a > 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\dfrac{u_{j+1}^n - u_j^n}{\Delta x} = 0\,, \text{ if } a < 0. \end{cases} \tag{5}$$

- *Lax-Friedrichs*

$$\frac{2u_j^{n+1} - u_{j+1}^n - u_{j-1}^n}{2\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0. \tag{6}$$

- *Beam-Warming* (if $a > 0$)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\Delta x} - \frac{a^2\Delta t}{2}\,\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} = 0. \tag{7}$$

Q1: We assume that u_0 is a periodic function. Unlike the other schemes, the *centered implicit* scheme does not allow, for a given space index j and a given time index n, to express explicitly u_j^{n+1} in function of the $(u_k^n)_k$. A linear system has to be solved. Construct the matrix of the linear system, prove it is invertible (let A be its matrix: show that $AU = 0 \Rightarrow U = 0$ by computing $U^t AU$). Show the L^2 stability unconditionally.

The *centered implicit* scheme can be written as

$$u_j^{n+1} + \frac{\nu}{2}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n\,,$$

for $j = 0 \ldots J$, and the convention that $u_{-1}^n = u_J^n$ and $u_{J+1} = u_0^n$. Explictly this is given by

$$\begin{pmatrix} u_0^n + \frac{\nu}{2}\left(u_1^{n+1} - u_J^{n+1}\right) \\ u_1^n + \frac{\nu}{2}\left(u_2^{n+1} - u_0^{n+1}\right) \\ u_2^n + \frac{\nu}{2}\left(u_3^{n+1} - u_1^{n+1}\right) \\ \vdots \\ u_J^n + \frac{\nu}{2}\left(u_0^{n+1} - u_{J-1}^{n+1}\right) \end{pmatrix} = \begin{pmatrix} u_0^n \\ u_1^n \\ u_2^n \\ \vdots \\ u_J^n \end{pmatrix},$$

which can be written as

$$A\boldsymbol{U}^{n+1} = \boldsymbol{U}^n\,,$$

with

$$\boldsymbol{U}^n = \begin{pmatrix} u_0^n \\ u_1^n \\ u_2^n \\ \vdots \\ u_J^n \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & \frac{\nu}{2} & 0 & 0 & \frac{-\nu}{2} \\ \frac{-\nu}{2} & 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & \frac{\nu}{2} \\ \frac{\nu}{2} & 0 & 0 & \frac{-\nu}{2} & 1 \end{pmatrix}.$$

One way to prove that the matrix A is invertible is to compute

$$\begin{aligned} \boldsymbol{U} \cdot A\boldsymbol{U} &= \left[u_0^2 + \frac{\nu}{2}u_0\left(u_1 - u_J\right)\right] + \left[u_1^2 + \frac{\nu}{2}u_1\left(u_2 - u_0\right)\right] + \left[u_2^2 + \frac{\nu}{2}u_2\left(u_3 - u_1\right)\right] + \\ &\quad \cdots + \left[u_J^2 + \frac{\nu}{2}u_J\left(u_0 - u_{J-1}\right)\right] \\ &= u_0^2 + u_1^2 + u_2^2 + \cdots + u_J^2 \\ &= \boldsymbol{U} \cdot \boldsymbol{U}\,. \end{aligned}$$

Therefore, we can write

$$\boldsymbol{U}^{n+1} = A^{-1}\boldsymbol{U}^n\,,$$

and obtain

$$\left\|\boldsymbol{U}^{n+1}\right\|^2 = \boldsymbol{U}^{n+1} \cdot \boldsymbol{U}^{n+1} \overset{\text{(property of } A)}{=} \boldsymbol{U}^{n+1} \cdot A\boldsymbol{U}^{n+1} \overset{\text{(definition of the scheme)}}{=} \boldsymbol{U}^{n+1} \cdot \boldsymbol{U}^n \le \left\|\boldsymbol{U}^{n+1}\right\| \left\|\boldsymbol{U}^n\right\|\,.$$

Therefore

$$\sum_j |u_j^{n+1}|^2 = \left\|\boldsymbol{U}^{n+1}\right\|^2 \le \left\|\boldsymbol{U}^n\right\|^2 = \sum_j |u_j^n|^2\,,$$

and the *centered implicit* scheme is unconditionally L^2-stable.

Q2: A finite volume scheme for equation (1) can be written

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n}{\Delta x} = 0, \tag{8}$$

where $f_{j\pm\frac{1}{2}}^n$ denotes a numerical flux. We still denote $\nu = \frac{a\Delta x}{dt}$.
Check that the *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming* schemes can be seen as a finite volume scheme with

Lax-Wendroff	$f_{j+\frac{1}{2}}^n = u_j^n + \frac{1}{2}(1-\nu)(u_{j+1}^n - u_j^n)$
upwind	$f_{j+\frac{1}{2}}^n = u_j^n$
Lax-Friedrichs	$f_{j+\frac{1}{2}}^n = \dfrac{u_{j+1}^n + u_j^n}{2} - \dfrac{u_{j+1}^n - u_j^n}{2\nu}$
Beam-Warming	$f_{j+\frac{1}{2}}^n = u_j^n + \frac{1}{2}(1-\nu)(u_j^n - u_{j-1}^n)$

This follows from simple algebraic calculations, for exemple for the *Lax-Wendroff* scheme:

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n}{\Delta x} &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{\Delta x}\left[u_j^n + \frac{1}{2}(1-\nu)(u_{j+1}^n - u_j^n) - u_{j-1}^n - \frac{1}{2}(1-\nu)(u_j^n - u_{j-1}^n)\right] \\ &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{a^2\Delta t}{2\Delta x}\left(2u_j^n - u_{j+1}^n - u_{j-1}^n\right) \\ &= 0\,. \end{aligned}$$

We sum up in the table below some properties of each scheme :

scheme	stability	truncation error
Lax-Wendroff	L^2 stable under CFL $\lvert a\rvert\Delta t \le \Delta x$ $[L^\infty$ stable if $\lvert a\rvert\Delta t = \Delta x]$	$O\big((\Delta t)^2 + (\Delta x)^2\big)$
centered explicit	unstable	$O\big(\Delta t + (\Delta x)^2\big)$
centered implicit	unconditionally L^2 stable	$O\big(\Delta t + (\Delta x)^2\big)$
upwind	L^2 and L^∞ stable under CFL $\lvert a\rvert\Delta t \le \Delta x$	$O(\Delta t + \Delta x)$
Lax-Friedrichs	L^2 and L^∞ stable under CFL $\lvert a\rvert\Delta t \le \Delta x$	$O\left(\Delta t + \dfrac{(\Delta x)^2}{\Delta t}\right)$
Beam-Warming	L^2 stable under CFL $\lvert a\rvert\Delta t \le 2\Delta x$	$O\big((\Delta t)^2 + (\Delta x)^2\big)$

Q3: Do you see one advantage to use the *Beam-Warming* scheme?

One advantage, is that the CFL is twice bigger, allowing bigger time steps.

Q4: For the following schemes: *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming*, show that if $a\Delta t = \Delta x$, the numerical solution u_j^n is equal to the analytical solution at the discretization point (x_j, t_n).

For the *Lax-Wendroff* scheme, one has $\alpha = \beta = 0$ and $\gamma = 1$ for $\nu = 1$, so

$$u_j^{n+1} = u_{j-1}^n\,.$$

One find the same result at $\nu = 1$ for all the listed schemes.
The exact solution is given by

$$\bar{u}(x,t) = u_0(x - at)\,,$$

so the evaluation of the exact solution at the discretization points gives

$$\begin{aligned} \bar{u}_j^{n+1} &= \bar{u}(x_j, t_{n+1}) = u_0(x_j - at_{n+1}) = u_0(x_j - a\Delta t - at_n) \\ &= u_0(x_j - \Delta x - at_n) = u_0(x_{j-1} - at_n) = \bar{u}(x_{j-1}, t_n) \\ &= \bar{u}_{j-1}^n\,, \end{aligned}$$

since $a\Delta t = \Delta x$. Since the scheme and the evalution of the exact solutions given the same relation, all the listed sheme are exact for $\nu = 1$. The reason is that the space-time grid is aligned with the characteristics, which are straight lines in this case.

Q5: By using the same tools as the ones used for the *Lax-Wendroff* scheme in section one, for each scheme of the table above, check its stability properties and its truncation error.

This can be done in the same way as for the *Lax-Wendroff* scheme.

Q6: Assuming $a > 0$, we introduce the third order scheme,

$$O3 = (1-\delta)LW + \delta BW\,, \quad \delta = \frac{1+\nu}{3} \tag{9}$$

where LW denotes the Lax-Wendroff scheme and BW denotes the Beam-Warming scheme. Check that this scheme is of order 3 in space and in time.

One has to perform the Taylor expansions to one order further to what has been done in section one.