

For a given $a \in \mathbb{R}$, we consider the following linear transport equation in one dimension :

$$\begin{cases} \partial_t \bar{u} + a \partial_x \bar{u} = 0, & \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+^+, \\ \bar{u}(x, 0) = u_0(x), & \forall x \in \mathbb{R}, \end{cases} \quad (1)$$

with $u_0 \in L^\infty(\mathbb{R})$. Without loss of generality, we assume that $a > 0$. We refer to the chapter 2, subsection 2.2.1, for the continuous framework of this equation. Here we focus on finding u a discrete approximation of \bar{u} thanks to discrete schemes. As in chapter 3, we introduce a discretization of the domain using a regular mesh : $(x_j, t_n) = (j\Delta x, n\Delta t)$, $\forall j \in \mathbb{Z}$, $\forall n \in \mathbb{N}$, where Δx , respectively Δt , denotes the space step, respectively the time step. We also denote u_j^n the approximation of $\bar{u}(x_j, t_n)$.

Definition: A scheme is L^∞ stable if we can prove the estimate

$$\sup_j |u_j^{n+1}| \leq \sup_j |u_j^n|.$$

Definition: A scheme is L^2 stable if we can prove the estimate

$$\sum_j |u_j^{n+1}|^2 \leq \sum_j |u_j^n|^2.$$

1 Lax-Wendroff scheme

We first focus on the *Lax-Wendroff* scheme :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2 \Delta t}{2} \frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} = 0. \quad (2)$$

Q1: Truncation error

The exact solution \bar{u} of (1) is generally not a solution of the scheme (2). The truncation error estimates the difference. Let us assume that the solution of (1) is such that $\bar{u} \in C^3(\mathbb{R} \times \mathbb{R}_+^+)$.

1. Prove that, for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+^+$, $\partial_{tt} \bar{u} = a^2 \partial_{xx} \bar{u}$.
2. Compute the Taylor expansions (“développements limités avec reste de Taylor-Lagrange”) at a convenient order of $\bar{u}(x_j, t_{n+1})$, $\bar{u}(x_{j+1}, t_n)$, and $\bar{u}(x_{j-1}, t_n)$ at the point (x_j, t_n) .
3. Assuming that enough partial derivatives of \bar{u} are bounded in L^∞ norm by some constant $C \in \mathbb{R}_+^+$, prove that the absolute value of the truncation error of the Lax-Wendroff scheme is second order both in time and space.

Q2: L^∞ stability

1. Show that, for any non-negative values α, β, γ such that $\alpha + \beta + \gamma = 1$, then

$$\forall x, y, z \in \mathbb{R}, \min(x, y, z) \leq \alpha x + \beta y + \gamma z \leq \max(x, y, z).$$

2. Using (2), find α, β, γ such that $u_j^{n+1} = \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n$.
3. Provide a necessary and sufficient condition on Δt , Δx and a ensuring the non-negativity of the coefficients α, β, γ found at the previous question. Deduce the L^∞ stability domain of the scheme.

Q3: L^2 stability

1. Show that

$$\sum_j |u_j^{n+1}|^2 = \sum_j |u_j^n|^2 - \frac{\nu^2(1-\nu^2)}{4} \sum_j |w_{j+1}^n - w_j^n|^2,$$

where $\nu = \frac{a\Delta t}{\Delta x}$ and $w_j^n = u_j^n - u_{j-1}^n$.

2. Deduce the condition under which the scheme is L^2 stable.

2 Schemes overview

- *Centered explicit scheme*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0. \quad (3)$$

- *Centered implicit scheme*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0. \quad (4)$$

- *Upwind scheme*

$$\begin{cases} \frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, & \text{if } a > 0, \\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0, & \text{if } a < 0. \end{cases} \quad (5)$$

- *Lax-Friedrichs*

$$\frac{2u_j^{n+1} - u_{j+1}^n - u_{j-1}^n}{2\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0. \quad (6)$$

- *Beam-Warming* (if $a > 0$)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\Delta x} - \frac{a^2 \Delta t}{2} \frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} = 0. \quad (7)$$

Q1: We assume that u_0 is a periodic function. Unlike the other schemes, the *centered implicit* scheme does not allow, for a given space index j and a given time index n , to express explicitly u_j^{n+1} in function of the $(u_k^n)_k$. A linear system has to be solved. Construct the matrix of the linear system, prove it is invertible (let A be its matrix: show that $AU = 0 \Rightarrow U = 0$ by computing $U^t A U$). Show the L^2 stability unconditionally.

Q2: A finite volume scheme for equation (1) can be written

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n}{\Delta x} = 0, \quad (8)$$

where $f_{j\pm\frac{1}{2}}^n$ denotes a numerical flux. We still denote $\nu = \frac{a\Delta x}{\Delta t}$.

Check that the *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming* schemes can be seen as a finite volume scheme with

<i>Lax-Wendroff</i>	$f_{j+\frac{1}{2}}^n = u_j^n + \frac{1}{2}(1-\nu)(u_{j+1}^n - u_j^n)$
<i>upwind</i>	$f_{j+\frac{1}{2}}^n = u_j^n$
<i>Lax-Friedrichs</i>	$f_{j+\frac{1}{2}}^n = \frac{u_{j+1}^n + u_j^n}{2} - \frac{u_{j+1}^n - u_j^n}{2\nu}$
<i>Beam-Warming</i>	$f_{j+\frac{1}{2}}^n = u_j^n + \frac{1}{2}(1-\nu)(u_j^n - u_{j-1}^n)$

We sum up in the table below some properties of each scheme :

scheme	stability	truncation error
<i>Lax-Wendroff</i>	L^2 stable under CFL $ a \Delta t \leq \Delta x$ [L^∞ stable if $ a \Delta t = \Delta x$]	$O((\Delta t)^2 + (\Delta x)^2)$
<i>centered explicit</i>	unstable	$O(\Delta t + (\Delta x)^2)$
<i>centered implicit</i>	unconditionally L^2 stable	$O(\Delta t + (\Delta x)^2)$
<i>upwind</i>	L^2 and L^∞ stable under CFL $ a \Delta t \leq \Delta x$	$O(\Delta t + \Delta x)$
<i>Lax-Friedrichs</i>	L^2 and L^∞ stable under CFL $ a \Delta t \leq \Delta x$	$O\left(\Delta t + \frac{(\Delta x)^2}{\Delta t}\right)$
<i>Beam-Warming</i>	L^2 stable under CFL $ a \Delta t \leq 2\Delta x$	$O((\Delta t)^2 + (\Delta x)^2)$

Q3: Do you see one advantage to use the *Beam-Warming* scheme?

Q4: For the following schemes: *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming*, show that if $a\Delta t = \Delta x$, the numerical solution u_j^n is equal to the analytical solution at the discretization point (x_j, t_n) .

Q5: By using the same tools as the ones used for the *Lax-Wendroff* scheme in section one, for each scheme of the table above, check its stability properties and its truncation error.

Q6: Assuming $a > 0$, we introduce the third order scheme,

$$O3 = (1 - \delta)LW + \delta BW, \quad \delta = \frac{1 + \nu}{3} \quad (9)$$

where LW denotes the Lax-Wendroff scheme and BW denotes the Beam-Warming scheme. Check that this scheme is of order 3 in space and in time.