Méthodes numériques pour les EDP instationnaires

TD 2: jeudi 23.09.2021
Stability and Fourier analysis

Solution

1 Modified equations

Q1: Determine the modified equation of the Lax-Wendroff sheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}+\frac{a^2\Delta t}{2}\frac{2u_j^n-u_{j-1}^n-u_{j+1}^n}{\Delta x^2}=0\,.$$

On the discretization of a function $\bar u$, we have

$$\bar u_j^{n+1}=\bar u(x_j,t_n+\Delta t)=\bar u(x_j,t_n)+\Delta t\,\partial_t\bar u(x_j,t_n)+\frac{\Delta t^2}{2}\partial_{tt}\bar u(x_j,t_n)+\frac{\Delta t^3}{6}\partial_{ttt}\bar u(x_j,t_n)+O(\Delta t^4)\,,$$
$$\bar u_{j+1}^{n}=\bar u(x_j+\Delta x,t_n)=\bar u(x_j,t_n)+\Delta x\,\partial_x\bar u(x_j,t_n)+\frac{\Delta x^2}{2}\partial_{xx}\bar u(x_j,t_n)+\frac{\Delta x^3}{6}\partial_{xxx}\bar u(x_j,t_n)+O(\Delta x^4)\,,$$
$$\bar u_{j-1}^{n}=\bar u(x_j-\Delta x,t_n)=\bar u(x_j,t_n)-\Delta x\,\partial_x\bar u(x_j,t_n)+\frac{\Delta x^2}{2}\partial_{xx}\bar u(x_j,t_n)-\frac{\Delta x^3}{6}\partial_{xxx}\bar u(x_j,t_n)+O(\Delta x^4)\,,$$

so we obtain

$$\frac{\bar u_j^{n+1}-\bar u_j}{\Delta t}=\partial_t\bar u(x_j,t_n)+\frac{\Delta t}{2}\partial_{tt}\bar u(x_j,t_n)+\frac{\Delta t^2}{6}\partial_{ttt}\bar u(x_j,t_n)+O(\Delta t^3)\,,$$
$$\frac{\bar u_{j+1}^n-\bar u_{j-1}^n}{2\Delta x}=\partial_x\bar u(x_j,t_n)+\frac{\Delta x^2}{6}\partial_{xxx}\bar u(x_j,t_n)+O(\Delta x^3)\,,$$
$$\frac{2\bar u_j^n-\bar u_{j-1}^n-\bar u_{j+1}^n}{\Delta x^2}=-\partial_{xx}\bar u(x_j,t_n)+O(\Delta x^2)\,.$$

Therefore the truncation error is

$$\begin{aligned}T_j^n&=\frac{\bar u_j^{n+1}-\bar u_j^n}{\Delta t}+a\frac{\bar u_{j+1}^n-\bar u_{j-1}^n}{2\Delta x}+\frac{a^2\Delta t}{2}\frac{2\bar u_j^n-\bar u_{j-1}^n-\bar u_{j+1}^n}{\Delta x^2}\\&=\partial_t\bar u(x_j,t_n)+\frac{\Delta t}{2}\partial_{tt}\bar u(x_j,t_n)+\frac{\Delta t^2}{6}\partial_{ttt}\bar u(x_j,t_n)+a\partial_x\bar u(x_j,t_n)\\&\quad+a\frac{\Delta x^2}{6}\partial_{xxx}\bar u(x_j,t_n)-\frac{\Delta t}{2}a^2\partial_{xx}\bar u(x_j,t_n)+O(\Delta t^3)+O(\Delta x^3)+O(\Delta x^2\Delta t)\\&=\partial_t\bar u(x_j,t_n)+a\partial_x\bar u(x_j,t_n)+\frac{\Delta t}{2}\left[\partial_{tt}\bar u(x_j,t_n)-a^2\partial_{xx}\bar u(x_j,t_n)\right]\\&\quad+\frac{\Delta x^2}{6a^2}\left[\nu^2\partial_{ttt}\bar u(x_j,t_n)+a^3\partial_{xxx}\bar u(x_j,t_n)\right]+O(\Delta t^3+\Delta x^3)\,.\end{aligned}$$

If $\bar u$ is a solution of $\partial_t\bar u+a\partial_x\bar u=0$, then $\partial_{tt}\bar u-a^2\partial_{xx}\bar u=0$ and $\partial_{ttt}\bar u+a^3\partial_{xxx}\bar u=0$, so this suggests that the modified equation is

$$\partial_t\bar u+a\partial_x\bar u+\frac{a\Delta x^2}{6}\left(1-\nu^2\right)\partial_{xxx}\bar u=0\,.$$

If $\bar u$ is a solution of the modified equation, one has

$$\partial_{tt}\bar u+a\partial_{tx}\bar u+\frac{a\Delta x^2}{6}\left(1-\nu^2\right)\partial_{txxx}\bar u=0\,,\qquad\partial_{tx}\bar u+a\partial_{xx}\bar u+\frac{a\Delta x^2}{6}\left(1-\nu^2\right)\partial_{xxxx}\bar u=0\,,$$

so

$$\partial_{tt}\bar u-a^2\partial_{xx}\bar u=\frac{a\Delta x^2}{6}\left(1-\nu^2\right)\partial_{txxx}\bar u+\frac{a^2\Delta x^2}{6}\left(1-\nu^2\right)\partial_{xxxx}\bar u=O(\Delta x^2)\,.$$

Therefore, on a solution of the modified equation, the truncation error is

$$T_j^n=O(\Delta t^3+\Delta x^3)\,,$$

which proves that the modified equation is correct.

Q2: Determine the modified equation of the Beam-Warming scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{3u_j^n-4u_{j-1}^n+u_{j-2}^n}{2\Delta x}-\frac{a^2\Delta t}{2}\frac{u_j^n-2u_{j-1}^n+u_{j-2}^n}{\Delta x^2}=0\,.$$

On the discretization of a function $\bar u$, the truncation error is

$$\begin{aligned}T_j^n&=\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{3u_j^n-4u_{j-1}^n+u_{j-2}^n}{2\Delta x}-\frac{a^2\Delta t}{2}\frac{u_j^n-2u_{j-1}^n+u_{j-2}^n}{\Delta x^2}\\&=\partial_t\bar u(x_j,t_n)+a\partial_x\bar u(x_j,t_n)+\frac{\Delta t}{2}\left[\partial_{tt}\bar u(x_j,t_n)-a^2\partial_{xx}\bar u(x_j,t_n)\right]\\&\quad+\frac{\Delta x^2}{6a^2}\left[\nu^2\partial_{ttt}\bar u(x_j,t_n)+a^3\left(3\nu-2\right)\partial_{xxx}\bar u(x_j,t_n)\right]+O(\Delta t^3+\Delta x^3)\,.\end{aligned}$$

Therefore, one can prove easily that the modified equation is

$$\partial_t\bar u+a\partial_x\bar u-\frac{a\Delta x^2}{6}\left(2-3\nu+\nu^2\right)\partial_{xxx}\bar u=0\,.$$

2 Stability and consistency

Q1: Show that the Beam-Warming scheme is L^2-stable for $\nu\le 2$ and second order in time and space.

The Beam-Warming scheme for transport is

$$u_j^{n+1}=u_j-\nu\frac{3u_j^n-4u_{j-1}^n+u_{j-2}^n}{2}+\nu^2\frac{u_j^n-2u_{j-1}^n+u_{j-2}^n}{2}\,,$$

where $\nu=\frac{a\Delta t}{\Delta x}$. The scheme can be rewritten as

$$u_j^{n+1}=\alpha u_j^n+\beta u_{j-1}^n+\gamma u_{j-2}^n\,,$$

where,

$$\alpha=1-\frac{3}{2}\nu+\frac{\nu^2}{2}\,,\qquad\beta=2\nu-\nu^2\,,\qquad\gamma=\frac{-\nu}{2}+\frac{\nu^2}{2}\,.$$

so the symbol of the scheme is

$$\lambda_h(\theta)=\alpha+\beta\mathrm{e}^{-\mathrm{i}\theta}+\gamma\mathrm{e}^{-2\mathrm{i}\theta}\,.$$

Stability: The scheme is L^2-stable if and only if

$$\sup_\theta|\lambda_h(\theta)|^2\le 1\,.$$

So we compute

$$|\lambda_h(\theta)|^2=\alpha^2+\beta^2+\gamma^2+2\beta(\alpha+\gamma)\cos\theta+2\alpha\gamma\cos(2\theta)\,,$$

and its derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}|\lambda_h(\theta)|^2&=-2\beta(\alpha+\gamma)\sin\theta-4\alpha\gamma\sin(2\theta)\\&=2\nu(\nu-2)(1-\nu)^2\sin\theta\left(1-\cos\theta\right)\,,\end{aligned}$$

is zero when $\theta=0$ or $\theta=\pi$ on $(-\pi,\pi]$. Therefore

$$\sup_\theta|\lambda_h(\theta)|^2=\max\left\{|\lambda_h(0)|^2,|\lambda_h(\pi)|^2\right\}=\max\left\{1,(1-2\nu(2-\nu))^2\right\}\,,$$

and the scheme is L^2-stable only when $\nu\in[0,2]$.
Consistency: The symbol of the transport operator being $\mu(\theta)=-\mathrm{i}a\theta$, one has

$$T=\left|\mathrm{e}^{\mu(\theta)\Delta t}-\lambda_h(\underbrace{\theta\Delta x}_{\bar\theta})\right|=\left|\mathrm{e}^{-\mathrm{i}\nu\bar\theta}-\lambda_h(\bar\theta)\right|=\left|\mathrm{e}^{-\mathrm{i}\nu\bar\theta}-\alpha-\beta\mathrm{e}^{-\mathrm{i}\bar\theta}-\gamma\mathrm{e}^{-2\mathrm{i}\bar\theta}\right|\,,$$

so by taken the Taylor expansion in $\bar\theta$, there exists a constant $C>0$ independent of ν, θ, Δt, and Δx such that

$$\begin{aligned}T&\le\left|1-\mathrm{i}\nu\bar\theta-\tfrac{\nu^2\bar\theta^2}{2}-\alpha-\beta+\mathrm{i}\beta\bar\theta+\beta\tfrac{\bar\theta^2}{2}-\gamma+2\mathrm{i}\gamma\bar\theta+2\gamma\bar\theta^2\right|+C\left(\nu^3|\bar\theta|^3+|\beta||\bar\theta|^3+|\gamma||\bar\theta|^3\right)\\&\le\left|1-\alpha-\beta-\gamma\right|+\left|\beta+2\gamma\right||\bar\theta|+\left|-\tfrac{\nu^2}{2}+\tfrac{\beta}{2}+2\gamma\right||\bar\theta|^2+C\left(\nu^3+|\beta|+|\gamma|\right)|\bar\theta|^3\,.\end{aligned}$$

Taking the definitions of α, β, and γ leads to the vanishing of the three first terms, so up to the modification of the constant C,

$$T\le C\left(\nu+\nu^3\right)|\bar\theta|^3\le C|\theta|^3\left(\Delta t^2+\Delta x^2\right)\Delta t\,.$$

This proves that the scheme is second order in space and time, therefore by the Lax theorem it is convergent.

Q2: Show that the Fourier symbol of Lax-Friedrichs scheme is not consistent (for small Δt).

The Lax-Friedrichs scheme is

$$u_j^{n+1}=\frac{u_{j+1}^n+u_{j-1}^n}{2}-\nu\frac{u_{j+1}^n-u_{j-1}^n}{2}\,,$$

where $\nu=\frac{a\Delta t}{\Delta x}$, so

$$u_j^{n+1}=\alpha u_{j-1}^n+\beta u_{j+1}^n\,,$$

where

$$\alpha=\frac{1}{2}-\frac{\nu}{2}\,,\qquad\beta=\frac{1}{2}+\frac{\nu}{2}\,.$$

The symbol of the scheme is

$$\lambda_h(\theta)=\alpha\mathrm{e}^{\mathrm{i}\theta}+\beta\mathrm{e}^{-\mathrm{i}\theta}\,.$$

For the consistency, there exists a constant $C>0$ such that

$$\begin{aligned}T&=\left|\mathrm{e}^{-\mathrm{i}\nu\bar\theta}-\lambda_h(\bar\theta)\right|\le\left|1-\mathrm{i}\nu\bar\theta-\left(\tfrac12-\tfrac{\nu}{2}\right)-\left(\tfrac12-\tfrac{\nu}{2}\right)\mathrm{i}\bar\theta-\left(\tfrac12+\tfrac{\nu}{2}\right)+\left(\tfrac12+\tfrac{\nu}{2}\right)\mathrm{i}\bar\theta\right|+C\left(\nu^2\bar\theta^2+\bar\theta^2\right)\\&\le C\left(\nu^2\bar\theta^2+\bar\theta^2\right)\le C\theta^2\left(\Delta t+\frac{\Delta x^2}{\Delta t}\right)\Delta t,\end{aligned}$$

and therefore one could guess the inconsistency of the scheme for small Δt. To make a proof, taking $\nu=0$ or $\Delta t=0$, one has

$$\lambda_h(\theta)=\cos\theta\,,$$

and for this specific value of ν,

$$T=\left|\mathrm{e}^{-\mathrm{i}\nu\bar\theta}-\lambda_h(\bar\theta)\right|=\left|1-\cos(\bar\theta)\right|\ge\frac{|\bar\theta|^2}{6}=\frac{1}{6}|\theta|^2\Delta x^2\,,$$

for $|\bar\theta|\le\pi$. Therefore T cannot be bounded by

$$\left(1+|\theta|^r\right)\left(\Delta t^p+\Delta x^q\right)\Delta t\,,$$

for any value of $p,q,r\ge 0$, and this proves that the scheme is not consistent for small values of Δt.

Q3: Study the 3-points scheme for diffusion

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2},\quad n\in\mathbb N,\ j\in\mathbb Z.$$

The scheme can be rewritten as

$$\begin{aligned}u_j^{n+1}&=u_j^n+\nu\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)\,,\\&=\left(1-2\nu\right)u_j^n+\nu u_{j+1}^n+\nu u_{j-1}^n\,,\end{aligned}$$

for $\nu=\frac{\Delta t}{\Delta x^2}$. The symbol of the scheme is

$$\lambda_h(\theta)=(1-2\nu)+\nu\mathrm{e}^{\mathrm{i}\theta}+\nu\mathrm{e}^{-\mathrm{i}\theta}=(1-2\nu)+2\nu\cos\theta\,.$$

For the stability, one directly find that

$$\sup_\theta|\lambda_h(\theta)|^2=\max\left\{1,1-4\nu\right\}\,,$$

so the scheme is L^2-stable if and only if $\nu\le\frac14$.
For the consistency, the symbol of the heat operator is $\mu(\theta)=-\theta^2$, and one can find the existence of a constant $C>0$ such that

$$T=\left|\mathrm{e}^{\mu(\theta)\Delta t}-\lambda_h(\underbrace{\theta\Delta x}_{\bar\theta})\right|=\left|\mathrm{e}^{-\nu\bar\theta^2}-\lambda_h(\bar\theta)\right|\le C\left(\nu+\nu^4\right)|\bar\theta|^4\le C|\theta|^4\left(\Delta t+\Delta x^2\right)\Delta t\,.$$

This proves that this scheme is first order in time and second order in space.

Q4: Study the 3-points scheme (à la Dufort-Frankl) for diffusion

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{u_{j+1}^n-2u_j^{n+1}+u_{j-1}^n}{\Delta x^2},\quad n\in\mathbb N,\ j\in\mathbb Z.$$

This implicit scheme can be rewritten as

$$u_j^{n+1}=u_j^n+\nu\left(u_{j+1}^n-2u_j^{n+1}+u_{j-1}^n\right)\,,$$

but also as an explicit scheme

$$u_j^{n+1}=\frac{1}{1+2\nu}u_j^n+\frac{\nu}{1+2\nu}u_{j+1}^n+\frac{\nu}{1+2\nu}u_{j-1}^n\,.$$

We remark that this scheme is L^∞-stable unconditionally. The symbol of the scheme is

$$\lambda_h(\theta)=\frac{1}{1+2\nu}+\frac{\nu}{1+2\nu}\mathrm{e}^{\mathrm{i}\theta}+\frac{\nu}{1+2\nu}\mathrm{e}^{-\mathrm{i}\theta}=\frac{1}{1+2\nu}\left(1+2\nu\cos\theta\right)\,,$$

and one has

$$\sup_\theta|\lambda_h(\theta)|^2=\max\left\{1,\frac{1-2\nu}{1+2\nu}\right\}\le 1\,,$$

so the scheme is L^2-stable unconditionally.
However, for the consistency, there exists a constant $C>0$ such that

$$\begin{aligned}T&=\left|\mathrm{e}^{-\nu\bar\theta^2}-\lambda_h(\bar\theta)\right|\ge\left|1-\nu\bar\theta^2-\tfrac{1}{1+2\nu}\left(1+2\nu-2\nu\bar\theta^2\right)\right|-C\left(\nu^2|\bar\theta|^4+\tfrac{\nu}{1+2\nu}|\bar\theta|^4\right)\\&\ge\frac{2\nu^2\bar\theta^2}{1+2\nu}-C(\nu+\nu^2)|\bar\theta|^4=\frac{2|\theta|^2}{1+2\frac{\Delta x^2}{\Delta t}}\Delta t-C|\theta|^4(\Delta x^2+\Delta t)\Delta t\,,\end{aligned}$$

and therefore the scheme is not consistent for small values of Δt.

3 Maximum principle for advection

Show that a second order (space-time) linear explicit scheme cannot preserve the maximum principle for all CFL number.
Hint: Show that second order polynomials are exactly evolved with such a scheme.

This is a theorem by Sergei Godunov, see for example Theorem 2 of the Wikipedia page *Godunov's theorem*.

4 Functional analysis in Fourier

This exercise can be considered as a pedestrian introduction to a much broader topic, see for example the Chapter 7 on the Hille-Yosida theorem of the book *Functional Analysis, Sobolev Spaces and Partial Differential Equations* by H. Brezis. The aim is to give a meaning to the representation formula $u(t)=e^{At}u_0$ where A is an unbounded operator. Typically $A=-a\partial_x$ or $A=\partial_{xx}$. The issue is that

$$e^{At}=\sum_{n\ge0}A^nt^n$$

is meaningless for unbounded operators. One needs to construct such an object by an approximation procedure.
Q1: We consider the Fourier decomposition $v(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb R}\widehat v(\theta)e^{i\theta x}d\theta$ and the truncation in Fourier space

$$v_\lambda(x)=\frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda}\hat v(\theta)e^{i\theta x}d\theta,\quad\lambda>0.$$

Show that $\lim_{\lambda\to\infty}v_\lambda=v$ for $v\in L^2(\mathbb R)$.
Measure the difference in $L^2(\mathbb R)$ for $v\in H^s(\mathbb R)$, $s\ge1$.

We have

$$v(x)-v_\lambda(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb R\setminus[-\lambda,\lambda]}\hat v(\theta)e^{i\theta x}d\theta\,,$$

which converge pointwise (at fixed values of x) to zero, so $\lim_{\lambda\to\infty}v_\lambda=v$ pointwise. However, this is not showing the convergence in L^2.
To this end, we remark that

$$v_\lambda(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb R}\chi_{[-\lambda,\lambda]}(\theta)\hat v(\theta)e^{i\theta x}d\theta\,,$$

where $\chi_{[-\lambda,\lambda]}$ is the indicator function of the set $[-\lambda,\lambda]$. Using the Plancherel theorem, we have

$$\begin{aligned}\|v-v_\lambda\|_{L^2}^2&=\left\|\hat v-\chi_{[-\lambda,\lambda]}\hat v\right\|_{L^2}^2=\left\|\left(1-\chi_{[-\lambda,\lambda]}\right)\hat v\right\|_{L^2}^2\\&=\int_{\mathbb R}\left(1-\chi_{[-\lambda,\lambda]}\right)^2|\hat v(\theta)|^2d\theta=\int_{\mathbb R\setminus[-\lambda,\lambda]}|\hat v(\theta)|^2d\theta\,,\end{aligned}$$

which converges to zero as $\lambda\to\infty$. Therefore $\lim_{\lambda\to\infty}v_\lambda=v$ in L^2.
In addition, if $v\in H^r$, then by definition

$$\|v\|_{H^s}^2=\left\|\left(1+|\cdot|^2\right)^{s/2}\hat v\right\|_{L^2}^2=\int_{\mathbb R}\left(1+|\theta|^2\right)^s|\hat v(\theta)|^2d\theta\,,$$

so

$$\|v-v_\lambda\|_{L^2}^2=\int_{\mathbb R\setminus[-\lambda,\lambda]}\frac{\left(1+|\theta|^2\right)^s|\hat v(\theta)|^2}{\left(1+|\theta|^2\right)^s}d\theta\le\frac{1}{\left(1+\lambda^2\right)^s}\int_{\mathbb R\setminus[-\lambda,\lambda]}\left(1+|\theta|^2\right)^s|\hat v(\theta)|^2d\theta\le\frac{\|v\|_{H^s}^2}{\left(1+\lambda^2\right)^s}\,,$$

so $\|v-v_\lambda\|_{L^2}$ decays to zero like λ^{-s} in the limit $\lambda\to0$.

Q2: Define the space

$$X_\lambda=\left\{v\in L^2(\mathbb R),\ \widehat v(\theta)=0\text{ for }|\theta|>\lambda\right\}.$$

Show that

$$\|a\partial_x\|_{\mathcal L(X_\lambda)}\le a\lambda.$$

For $v\in X_\lambda$, one has

$$\|a\partial_xv\|_{L^2}=\|a\theta\hat v\|_{L^2}\le a\lambda\|\hat v\|_{L^2}\,,$$

so

$$\|a\partial_x\|_{\mathcal L(X_\lambda)}\le a\lambda.$$

Q3: Give a meaning in $\mathcal L(X_\lambda)$ to the series

$$e^{-a\partial_xt}=\sum_{n\ge0}\frac{1}{n!}(-a\partial_xt)^n.$$

Taking the L^2-norm, one has

$$\left\|e^{-a\partial_xt}v\right\|_{L^2}=\left\|\sum_{n\ge0}\frac{1}{n!}(-a\partial_xt)^nv\right\|_{L^2}\le\sum_{n\ge0}\frac{1}{n!}\|(-a\partial_xt)^nv\|_{L^2}\le\sum_{n\ge0}\frac{(a\lambda t)^n}{n!}\|v\|_{L^2}\le e^{a\lambda t}\|v\|_{L^2}\,,$$

so the series is absolutely convergent, hence convergent for $v\in X_\lambda$. Obviously $e^{-a\partial_xt}v\in X_\lambda$, so $e^{-a\partial_xt}$ defines an operator on X_λ.

Q4: Consider the problem ($a>0$)

$$\begin{cases}\partial_tu_\lambda+a\partial_xu_\lambda=0, & t>0,\quad x\in\mathbb R,\\ u_\lambda(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda}\hat u_0(\theta)e^{i\theta x}d\theta\end{cases}$$

where $\hat u_0\in L^2(\mathbb R)$. What does mean $u_\lambda(t)=e^{-a\partial_xt}u_\lambda(0)$? Show the a priori bound

$$\|u_\lambda(t)\|_{L^2(\mathbb R)}\le e^{a\lambda t}\|u_\lambda(0)\|_{L^2(\mathbb R)}.$$

In view of the previous question, we can define

$$u_\lambda(t)=e^{-a\partial_xt}u_\lambda(0)$$

which is in X_λ. Taking the time derivative leads to

$$\begin{aligned}\partial_tu_\lambda&=\sum_{n\ge1}\frac{1}{n!}nt^{n-1}(-a\partial_x)^nu_\lambda(0)=-a\sum_{n\ge1}\frac{1}{(n-1)!}(-a\partial_xt)^{n-1}\partial_xu_\lambda(0)\\&=-a\sum_{n\ge0}\frac{1}{n!}(-a\partial_xt)^n\partial_xu_\lambda(0)=-a\partial_xu_\lambda\,,\end{aligned}$$

which is still an absolutely convergent series. Therefore, $u_\lambda(t)$ is a solution of the equation. The bound follows from the previous question.

Q5: Show there exists $u\in L^\infty(0,T;L^2(\mathbb R))$ such that

$$\lim_{\lambda\to\infty}u_\lambda=u\text{ in }L^\infty(0,T;L^2(\mathbb R)).$$

The simplest way is to identify the limit with

$$u(t)=\mathcal F^{-1}\left(e^{-\mathrm{i}a\theta t}\hat u_0\right)\,,$$

which is a function in L^2 for each values of t. Since

$$\|u(t)\|_{L^2}^2=\left\|e^{-\mathrm{i}a\theta t}\hat u_0\right\|_{L^2}^2=\|\hat u_0\|_{L^2}^2=\|u_0\|_{L^2}^2\,,$$

this norm is $L^\infty(0,T)$, which means that $u\in L^\infty(0,T:L^2(\mathbb R))$. To proven the convergence, we have

$$\|u_\lambda(t)-u(t)\|_{L^2}^2=\|\hat u_\lambda(t)-\hat u(t)\|_{L^2}^2=\left\|e^{-\mathrm{i}a\theta t}\hat u_\lambda(0)-e^{-\mathrm{i}a\theta t}\hat u_0\right\|_{L^2}^2=\|\hat u_\lambda(0)-\hat u_0\|_{L^2}^2\to0$$

as $\lambda\to0$.

Q6: Generalize to the problem

$$\begin{cases}\partial_tu=\partial_{xx}u, & t>0,\quad x\in\mathbb R,\\ u_0\in L^2(\mathbb R), & x\in\mathbb R.\end{cases}$$

The proof goes exactly in the same spirit, except that we now have inequalities like

$$\|u(t)\|_{L^2}^2=\left\|e^{-\theta^2t}\hat u_0\right\|_{L^2}^2\le\|\hat u_0\|_{L^2}^2=\|u_0\|_{L^2}^2\,,$$

instead of equalities.

5 Weighted norm

The following example shows that the choice of the norm matters for the numerical analysis, specially if one takes weighted norms.

Take two positive parameters $a,\sigma>0$. Let u be the solution of the transport equation with a source

$$\begin{cases}\partial_tu+a\partial_xu=\sigma u, & x\in\mathbb R,\quad t>0,\\ u(0,x)=u_0(x), & x\in\mathbb R.\end{cases}$$

We will systematically assume that all solutions u tend sufficiently fast to zero at infinity.
Q1: Give the exact solution in function of the initial data.
Hint: one can constructs a formula for the solution under the form $u(t,x)=\varphi(t)u_0(x-at)$.

Let

$$u(t,x)=\varphi(t)u_0(x-at)\,,$$

for some unkown function φ. One has

$$\begin{aligned}\partial_tu&=\varphi'(t)u_0(x-at)-a\varphi(t)u_0'(x-at)\,,\\\partial_xu&=\varphi(t)u_0'(x-at)\,,\end{aligned}$$

so

$$\partial_tu+a\partial_xu=\varphi'(t)u_0(x-at)\,.$$

In order that u is a solution of the equation, the function φ has to solve

$$\varphi'(t)=\sigma\varphi(t)\,,$$

i.e.

$$\varphi(t)=e^{\sigma t}\,.$$

Q2: Show that $\|u(t)\|_{L^2(\mathbb R)}=e^{\sigma t}\|u_0\|_{L^2(\mathbb R)}$.

We have

$$\|u(t)\|_{L^2}=|\varphi(t)|\|u_0\|_{L^2}=e^{\sigma t}\|u_0\|_{L^2}$$

Q3: Determine the properties of the symbol of the operator.

The operator putting the equation under the form $\partial_tu=Au$ is

$$A=\sigma-a\partial_x\,,$$

so its symbol is

$$\mu(\theta)=\sigma-\mathrm{i}a\theta\,.$$

Q4: Consider the weighted norm

$$\|u\|=\|u\sqrt w\|_{L^2(\mathbb R)}=\sqrt{\int_{\mathbb R}u(x)^2w(x)dx}$$

where $w>0$ is the weight.
Find w such that

$$\|u(t)\|=\|u_0\|,\quad\forall t>0,$$

for all possible u_0.
Indication: by derivation of the criterion, one can try to construct an equation satisfied by w and then solve the equation.

This aim is to find a weight $w>0$ such that

$$\int_{\mathbb R}u(x,t)^2w(x)dx=\int_{\mathbb R}u_0(x)^2w(x)dx\,.$$

Taking the time derivative of the left rand side leads to

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathbb R}u(x,t)^2w(x)dx\right)&=2\int_{\mathbb R}u(x,t)\partial_tu(x,t)w(x)dx\\&=2\int_{\mathbb R}u(x,t)\left(\sigma u(x,t)-a\partial_xu(x,t)\right)w(x)dx\\&=2\int_{\mathbb R}\sigma u(x,t)^2w(x)dx-2a\int_{\mathbb R}u(x,t)\partial_xu(x,t)w(x)dx\\&=2\int_{\mathbb R}\sigma u(x,t)^2w(x)dx-a\int_{\mathbb R}\partial_x(u(x,t)^2)w(x)dx\\&=2\int_{\mathbb R}\sigma u(x,t)^2w(x)dx+a\int_{\mathbb R}u(x,t)^2w'(x)dx\\&=2\int_{\mathbb R}u(x,t)^2\left[2\sigma w(x)+aw'(x)\right]dx\,.\end{aligned}$$

Therefore, this expression is zero in general provided the weight satisfies

$$2\sigma w+aw'=0\,,$$

so the weight is

$$w(x)=e^{-\frac{2\sigma x}{a}}\,.$$

Q5: Consider the upwind scheme under the form

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{u_j^n-u_{j-1}^n}{\Delta x}=\sigma u_{j-1}^n$$

where the source is also upwinded. The numerical solution at time step $t_n=n\Delta t$ is $u_h^n=\left(u_j^n\right)_{j\in\mathbb Z}$ and its discrete norm is

$$\|u_h^n\|=\sqrt{\sum_j|u_j^n|^2w_j\Delta x},\qquad w_j=e^{-\frac{2\sigma}{a}j\Delta x}.$$

Assume the CFL condition $\nu=a\frac{\Delta t}{\Delta x}\le1$. Show the inequality

$$\|u^{n+1}\|\le Q(\sigma)\|u^n\|,\quad Q(\sigma)=1-\nu+(\nu+\sigma\Delta t)\,e^{-\frac{\sigma}{a}\Delta x}.$$

Indication: show that

$$\sum_j|u_j^{n+1}|^2w_j=Q(\sigma)^2\sum_j|u_j^n|^2w_j-(1-\nu)(\nu+\sigma\Delta t)\sum_j\left(u_j^n\sqrt{w_{j+1}}-u_{j-1}^n\sqrt{w_j}\right)^2.$$

The upwind scheme can be written as

$$u_j^{n+1}=\alpha u_j^n+\beta u_{j-1}^n\,,$$

with

$$\alpha=1-\nu\,,\qquad\beta=\nu+\sigma\Delta t\,.$$

Since

$$w_j=e^{-\frac{2\sigma}{a}j\Delta x}=e^{-\frac{2\sigma}{a}\Delta x}e^{-\frac{2\sigma}{a}(j-1)\Delta x}=e^{-\frac{2\sigma}{a}\Delta x}w_{j-1}\,,$$

one has for $\nu\le1$ that

$$\begin{aligned}\sum_j|u_j^{n+1}|^2w_j&=\sum_j\left(\alpha^2|u_j^n|^2+\beta^2|u_{j-1}^n|^2+2\alpha\beta u_j^nu_{j-1}^n\right)w_j\\&=\sum_j\left(\alpha^2|u_j^n|^2w_j+\beta^2|u_j^n|^2w_{j+1}+2\alpha\beta u_j^n\sqrt{w_j}u_{j-1}^n\sqrt{w_{j-1}}\mathrm{e}^{-\frac{\sigma}{a}\Delta x}\right)\\&=\sum_j\left(\alpha^2+\beta^2\mathrm{e}^{-\frac{2\sigma}{a}\Delta x}\right)|u_j^n|^2w_j-\alpha\beta\mathrm{e}^{-\frac{\sigma}{a}\Delta x}\sum_j\left(u_j^n\sqrt{w_j}-u_{j-1}^n\sqrt{w_{j-1}}\right)^2\\&\qquad+\alpha\beta\mathrm{e}^{-\frac{\sigma}{a}\Delta x}\sum_j\left(|u_j^n|^2w_j+|u_{j-1}^n|^2w_{j-1}\right)\\&=\sum_j\left(\alpha+\beta\mathrm{e}^{-\frac{\sigma}{a}\Delta x}\right)^2|u_j^n|^2w_j-\alpha\beta\mathrm{e}^{-\frac{\sigma}{a}\Delta x}\sum_j\left(u_j^n\sqrt{w_j}-u_{j-1}^n\sqrt{w_{j-1}}\right)^2\\&=\sum_j\left(\alpha+\beta\mathrm{e}^{-\frac{\sigma}{a}\Delta x}\right)^2|u_j^n|^2w_j-\alpha\beta\sum_j\left(u_j^n\sqrt{w_{j+1}}-u_{j-1}^n\sqrt{w_j}\right)^2\\&\le\sum_jQ(\sigma)^2|u_j^n|^2w_j\,,\end{aligned}$$

where

$$Q(\sigma)=\alpha+\beta\mathrm{e}^{-\frac{\sigma}{a}\Delta x}=1-\nu+(\nu+\sigma\Delta t)\,e^{-\frac{\sigma}{a}\Delta x}\,.$$

Therefore we just prove that

$$\|u^{n+1}\|\le Q(\sigma)\|u^n\|\,.$$

Q6: Show that $Q'(\sigma)\le0$ and prove the uniform stability (in the weighted discrete norm) of the upwind scheme.

One has

$$Q'(\sigma)=\Delta te^{-\frac{\sigma}{a}\Delta x}-\frac{\Delta x}{a}\left(\nu+\sigma\Delta t\right)e^{-\frac{\sigma}{a}\Delta x}=-\frac{\Delta t\Delta x\sigma}{a}e^{-\frac{\sigma}{a}\Delta x}\le0\,.$$

Since $Q(0)=1$, we have $Q(\sigma)\le1$ for $\sigma\ge0$, so the scheme is stable in the weighted norm for $\nu\le1$.
Another easiest way to prove the stability is to use the Fourier theory. We define

$$v_j^n=u_j^n\sqrt{w_j}\,,$$

so that

$$v_j^{n+1}=(1-\nu)v_j^n+(\nu+\sigma\Delta t)\,e^{-\frac{\sigma}{a}\Delta x}v_{j-1}^n=\alpha v_j^n+\bar\beta v_{j-1}^n\,,$$

where

$$\alpha=1-\nu\,,\qquad\bar\beta=\nu+\delta\,,\qquad\delta=\sigma\Delta te^{-\frac{\sigma}{a}\Delta x}+\nu\left(e^{-\frac{\sigma}{a}\Delta x}-1\right)\,.$$

Therefore, the symbol of the scheme is

$$\lambda_h(\theta)=\alpha+\bar\beta e^{-i\theta}\,,$$

so

$$|\lambda_h(\theta)|^2=\alpha^2+\bar\beta^2+2\alpha\bar\beta\cos\theta\,.$$

and therefore

$$\sup_\theta|\lambda_h(\theta)|^2=\max\left\{(\alpha+\bar\beta)^2,(\alpha-\bar\beta)^2\right\}=Q(\sigma)^2\le1\,,$$

which prove the L^2-stability of v_j^n *i.e.* the stability of u_j^n in the weighted norm.

Q7: Prove that the scheme is convergent (eventually under the simplified assumption that the support of the functions u_0 and u is compact in space).

The simplest way is probably to make the following change of variables

$$v_j^n=u_j^n\sqrt{w_j}\,,\qquad v(x,t)=u(x,t)\sqrt{w(x)}\,.$$

In view of the equation solved by v,

$$\partial_tv+a\partial_xv=0\,,$$

the symbol of the equation for v is

$$\mu(\theta)=-\mathrm{i}a\theta\,.$$

The symbol of the scheme for v_j^n was already deduce in the previous question, and since

$$\delta=\nu\left[\frac{\sigma}{a}\Delta xe^{-\frac{\sigma}{a}\Delta x}+e^{-\frac{\sigma}{a}\Delta x}-1\right]\,,$$

we observe that

$$|\delta|=\nu\left[\frac{\sigma}{a}\Delta x-\frac{\sigma}{a}\Delta x+O(\Delta x^2)\right]\le C\nu\Delta x^2\,.,$$

where $C>0$ depends only on a and σ. Therefore, there exists a constant $C>0$ independent of ν such that

$$\begin{aligned}T&=\left|\mathrm{e}^{\mu(\theta)\Delta t}-\lambda_h(\underbrace{\theta\Delta x}_{\bar\theta})\right|=\left|\mathrm{e}^{-\mathrm{i}\nu\bar\theta}-\lambda_h(\bar\theta)\right|=\left|\mathrm{e}^{-\mathrm{i}\nu\bar\theta}-\alpha+\nu e^{-i\theta}\right|+\left|\delta e^{-i\theta}\right|\\&\le C\left(\nu^2\bar\theta^2+\nu\bar\theta^2+\nu\Delta x^2\right)\le C\left(\Delta x\theta^2+\Delta t\theta^2+\Delta x\right)\Delta t\\&\le C\left(1+\theta^2\right)\left(\Delta t+\Delta x\right)\Delta t\,.\end{aligned}$$

This proves that the scheme for v_j^n is L^2-stable hence convergent, which means that the scheme in u_j^n is also convergent in the weighted norm.