Méthodes numériques pour les EDP instationnaires

TP 2: jeudi 30.09.2021
Transport equation with variable coefficients

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1 Preliminary materials

For a given real function $a \in C^1(\mathbb{R})$ such that $|a|$ and $|a'|$ are bounded on $\mathbb{R}$ by some $A > 0$, we consider the following linear transport equation in one dimension

$$\begin{cases} \partial_t u + a(x)\partial_x u = 0\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ u(x,0) = u_0(x)\,, & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u_0 \in C_c^1(\mathbb{R}) = C_0^1(\mathbb{R})$ has a compact support. This equation is not in divergence form, it is called the **non conservative equation**.

Denote by $y(X,t)$ the solution of

$$\begin{cases} \partial_t y(X,t) = a(y(X,t))\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}, \\ y(X,0) = X\,, & \forall X \in \mathbb{R}. \end{cases}$$

1. Show that $u(x,t) = u_0(X)$ with $x = y(X,t)$. What could mean the formula $u(x,t) = u_0(Y(x,t))$?

2. Next we consider

$$\begin{cases} \partial_t v + \partial_x (a(x)v) = 0\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ v(x,0) = v_0(x)\,, & \forall x \in \mathbb{R}. \end{cases} \tag{2}$$

   This equation is in divergence form, it is called the **conservative equation**.

   Show that $v(x,t) = J(X,t)v_0(X)$ with $J(X,t) = e^{-\int_0^t \partial_x a(y(X,s))ds} > 0$.

3. For $a(x) = x$, show that $v(x,t) = e^{-t}v_0(X)$.

2 Characteristics

The aim of this exercise is to get familiar with the characteristics for various velocity a.

1. Plot the characteristics of

$$x' = a(x) \qquad \text{with} \qquad a(x) = x.$$

   If you are not used to Python, use:
   `https://www.ljll.math.upmc.fr/despres/BD_fichiers/caracteristics.py`

2. Plot the characteristics in the cases, $a(x) = \pm x$, $a(x) = \sin(2\pi x)$.

3. Now we consider the case where the velocity/celerity is not Lipshitz continuous.
   Plot the characteristics for $a(x) = \pm\sqrt{|x|}$.

4. Plot the characteristics for $a(x) = \pm\text{sign}(x)$ and propose an interpretation.

3 Numerical schemes

The aim is to define numerical scheme for solving (1) and (2). For $a \in \mathbb{R}$, we define $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

1. The scheme
$$\Delta x \frac{u_j^{n+1} - u_j^n}{\Delta t} + a_j^-(u_j^n - u_{j+1}^n) - a_{j-1}^+(u_{j-1}^n - u_j^n) \;=\; 0.$$
can be used to solve (1).

   Implement this scheme on $[-1, 1]$ with periodic boundary conditions and test it for $a(x) = x$ and $u_0 = \chi_{[-0.2,0.2]}$.

   You can use: `https://www.ljll.math.upmc.fr/despres/BD_fichiers/transport.py`.

2. Write the analytical solution and compare it graphically with the discrete solution.

3. Implement the scheme
$$\Delta x \frac{u_j^{n+1} - u_j^n}{\Delta t} + (a_j^+ u_j^n - a_j^- u_{j+1}^n) - (a_{j-1}^+ u_{j-1}^n - a_{j-1}^- u_j^n) \;=\; 0$$
to solve (2).

4. Write in the new analytical solution and compare it graphically with the discrete solution.

5. Try to observe some of the scheme properties as the L^∞ stability, the discrete mass conservation and the discrete maximum principal.

6. Do the same work in the case $a(x) = -x$ and for a better understanding you can plot the characteristics on an other window.

7. Determine numerically the order of convergence of the schemes. Observe that the order of convergence depends heavily on the regularity of the initial condition and on the L^p-norm.