Mécanique II : Série 4

1 Systems with one degree of freedom

A system with a single degree of freedom is described by the differential equation

$$\ddot{x} = f(x) = -\frac{dU}{dx}\,,$$

and its conserved total energy is the sum

$$E = \frac{1}{2}\dot{x}^2 + U(x)\,.$$

1. Show that the time it takes to reach x_2 from x_1 is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2E - 2U(x)}}\,.$$

From the energy equation it follows that $\frac{dx}{dt} = \pm\sqrt{2E - 2U(x)}$. Integration leads to above result.

2. Consider the symmetric potentials $U(x) = \frac{1}{2n}x^{2n}$, $n \in \mathbb{N}$. For $E > 0$, the system performs a periodic motion (oscillation) between the two points $x = \pm a$ where $U(\pm a) = E$. How does the period depend on the amplitude a of the oscillation ?

From the previous result we know that

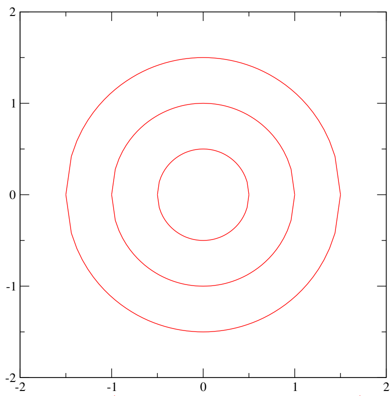
$$\frac{P}{4} = \int_0^a \frac{dx}{\sqrt{\frac{1}{n}a^{2n} - \frac{1}{n}x^{2n}}} = \int_0^a \frac{\sqrt{n}dx}{a^n\sqrt{1 - \left(\frac{x}{a}\right)^{2n}}} = a^{1-n}\int_0^1 \frac{\sqrt{n}ds}{\sqrt{1 - s^{2n}}}\,.$$

The last integral is independent of a, therefore we have $P \propto a^{1-n}$.

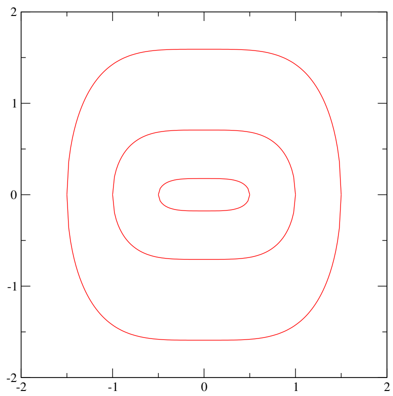
3. Is there a choice of n for the above potentials where the period is independent of the amplitude ?

Yes, $n = 1$. As is well known, the frequency of the harmonic oscillator is independent of the amplitude.

4. Sketch the *phase curves* for the cases $n = 1$ and $n = 2$.



$n = 1$ (harmonic oscillator)

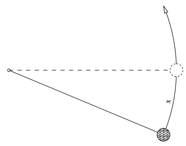


$n = 2$

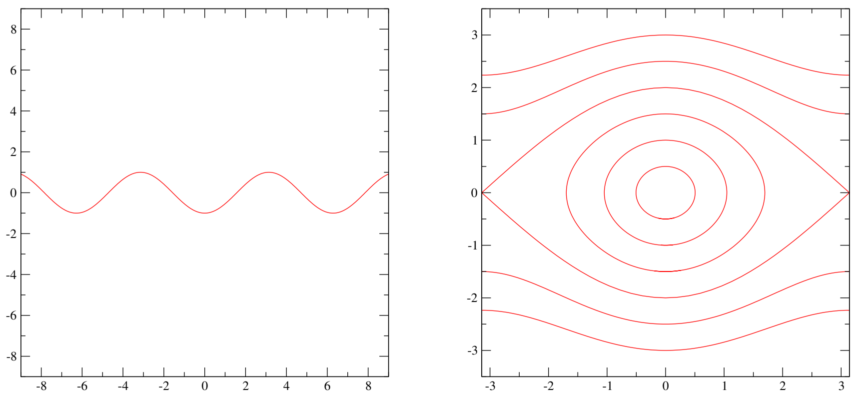
An ideal planar pendulum (see figure) is described by the equation

$$\ddot{x} = -\sin(x)\,.$$

Remark : coordinates which differ by an integer multiple of 2π correspond to the same position of the pendulum. In other words, each coordinate x is mapped to a point on a circle, and each point in the phase plane is mapped to a point on a *phase cylinder*. These maps are called *covers*, and the real line $\mathbb{R}$ and the phase plane are called *universal covering spaces* of the circle and the phase cylinder, respectively.



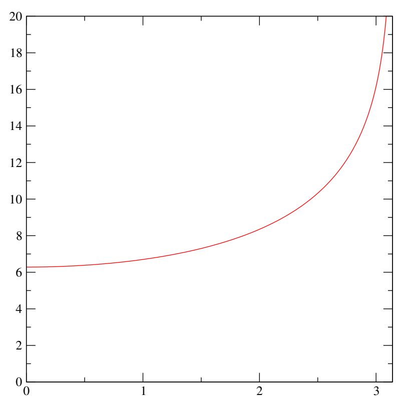
5. Sketch the potential energy and the phase curves for several choices of total energy. Discuss the three cases $E < U(\pi)$, $E = U(\pi)$, and $E > U(\pi)$. How many distinct phase curves are there for a fixed energy level E?



-1cm$U(x)$ = constant $-\cos(x)$. For $E < U(\pi)$ there is one phase curve per energy level (up to identifications). For $E = U(\pi)$ there are three distinct phase curves : one in the upper half plane going from $x = -\pi$ to $x = \pi$, one in the lower half plane going the opposite direction, and a single point at $x = \pm\pi$. These are the critical curves making up the separatrix. For $E > U(\pi)$ there are two distinct curves : one in the upper half plane for rotation in positive x-direction, and one in the lower half plane for opposite rotation.

6. Sketch the graph of $P(a)$, where P is the period of the periodic motion for given amplitude a. *Hint :* consider the behavior of P and dP/da when a approaches the two critical values $a \to 0$ and $a \to \pi$.

For $a \to 0$ we can set $\sin(x) \approx x$ and the pendulum behaves like a harmonic oscillator. Therefore $P \to 2\pi$ and $dP/da \to 0$. For $a \to \pi$ the period has to diverge, i.e. $P \to \infty$ and $dP/da \to \infty$.

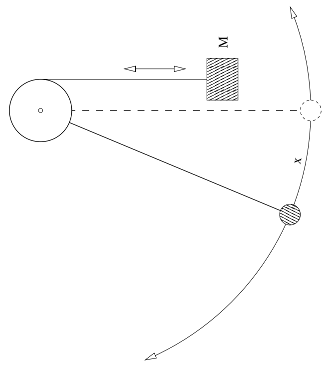


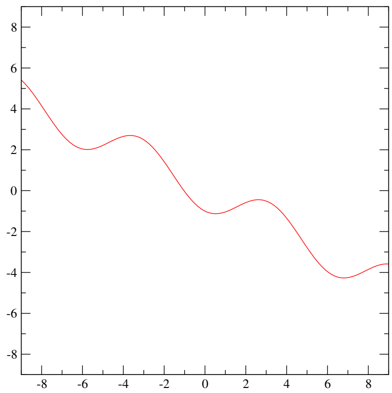
Let a constant torque M be applied to the pendulum. The equation of motion becomes

$$\ddot{x} = -\sin(x) + M\,.$$

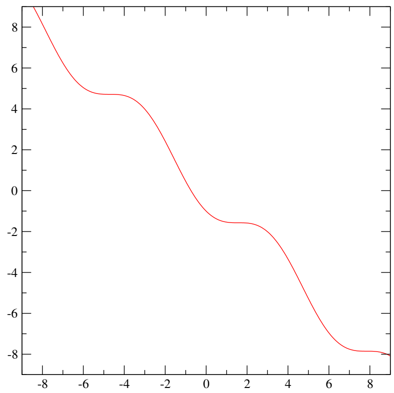
7. Sketch the potential energy for the three cases $0 < M < 1$, $M = 1$, and $M > 1$. Identify the *equilibrium points* where $x =$ constant, i.e. $\dot{x} = \ddot{x} = 0$. Which of them are stable?

   The equilibrium points are points on the potential curves where the slope is zero. Stable points are the local minima of the potential. They only exist in the case $0 < M < 1$.

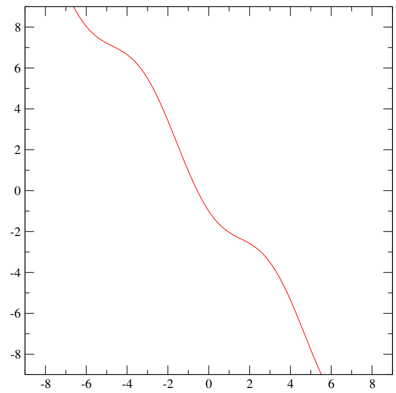




$0 < M < 1$

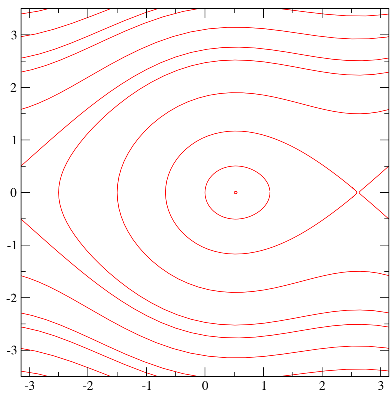


$M = 1$

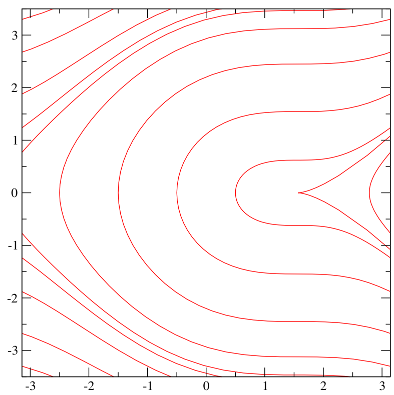


$M > 1$

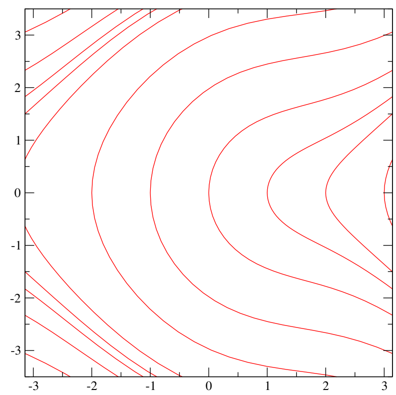
8. Sketch and discuss various possible phase curves for all three cases mentioned above. Where can one find periodic motion?



$0 < M < 1$



$M = 1$



$M > 1$

For $0 < M < 1$ there are periodic orbits around the stable equilibrium points. The equilibrium points correspond to configurations where the pendulum is raised from its vertical downward position by an arc length such that its torque exactly compensates the constant torque M, i.e. $\sin(x) = M$. This equation has two solutions for $0 < M < 1$, one of which is below the horizontal position (stable) and the other one being symmetrically above the horizontal (unstable). When $M \to 1$, these two solutions converge towards the horizontal position, which produces the maximum torque. For $M > 1$

the pendulum cannot compensate the external torque, and all orbits are unbound. Starting with some initial velocity in negative x-direction, the pendulum slows down while it rotates, eventually stops and reverses its sense of rotation, then speeds up indefinitely, rotating faster and faster.

2 Phase flow

Let $p(0)$ be a point in the phase plane. The phase curve passing through this point belongs to the solution with initial conditions at $t = 0$ given by $p(0)$. Assuming that the solution can be extended to the entire time axis, its value $p(t)$ is defined at any t and is uniquely determined by $p(0)$. We can introduce a map g^t from the phase space to itself by writing

$$p(t) = g^t p(0)\,.$$

This map is a *diffeomorphism*, i.e. it is bijective, differentiable, and has a differentiable inverse. The set of diffeomorphisms g^t, $t \in \mathbb{R}$ has an abelian group structure : the group operation is the composition $g^{t+s} = g^t \circ g^s = g^s \circ g^t$, the identity element is g^0, and g^{-t} is the inverse of g^t. One says that the transformations g^t form a *one-parameter group of diffeomorphisms* of the phase plane. This group is called the *phase flow*.

1. Show that the system with a single degree of freedom and with potential $U(x) = -x^4$ does not define a phase flow.

   A solution with initial conditions $x(0) = x_0 > 0$ and $\dot{x}(0) = 0$ reaches $x = \infty$ in finite time :

   $$t_\infty = \int_{x_0}^{\infty} \frac{dx}{\sqrt{2x^4 - 2x_0^4}} \approx \frac{0.927}{x_0}\,.$$

   Therefore, the solution cannot be extended to the entire time axis, and the map g^t is not defined on the phase point if $t \geq t_\infty$.

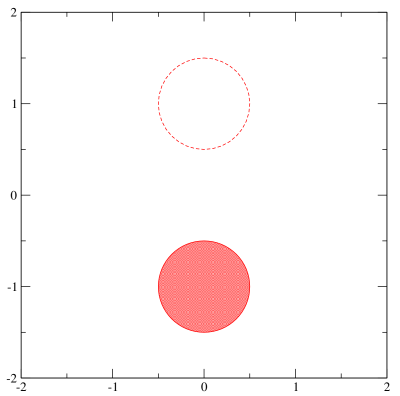
2. Show that for systems with a single degree of freedom, to define a phase flow, it is sufficient to have a potential energy which is bounded from below.

   From the conservation of energy we know that $|\dot{x}| = \sqrt{2E - 2U(x)}$. If $U(x)$ is bounded from below by $U_{\min}$, it follows that $E \geq U_{\min}$ and $|\dot{x}|$ is bounded from above by $\sqrt{2E - 2U_{\min}}$. If the velocity is bounded, the solution has to stay finite for any finite time – it can be extended to the entire time axis.

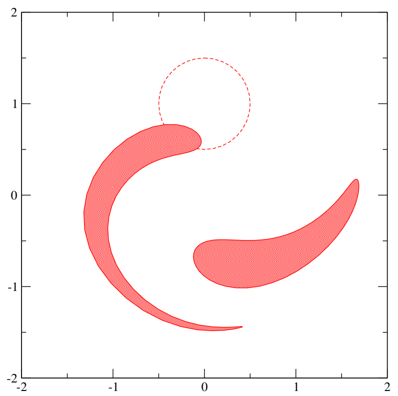
3. Sketch the image of a disc in phase space, defined by the condition $x^2 + (\dot{x} - 1)^2 < \frac{1}{4}$, under the action of an element of the phase flow for

   a) the harmonic oscillator $\ddot{x} = -x$, and

   b) the ideal planar pendulum $\ddot{x} = -\sin(x)$.



For the harmonic oscillator, the period does not depend on the initial conditions. Therefore, the disc is simply rotated around the origin with period 2π. The plot shows the disc at $t = \pi$ $(, 3\pi, 5\pi, \ldots)$.



For the ideal planar pendulum and in the phase space region where the disc is placed, the period increases with distance from the origin. Therefore, the disc gets gradually deformed, its outward pointing part lagging behind the inward pointing one. The plot shows the image of the disc for $t = 3\pi$ and $t = 6\pi$.

3 Systems with two degrees of freedom

A system with two degrees of freedom is described by the differential equation

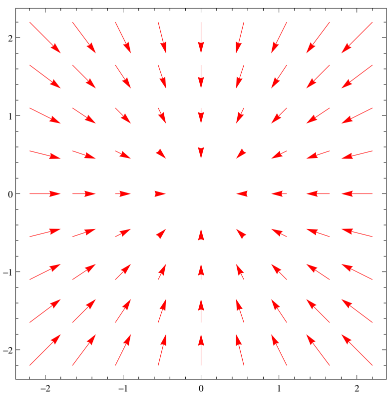
$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x}$ is a point on a two-dimensional manifold (e.g. the Euclidean plane) and $\mathbf{f}$ is a vector field on this manifold. The phase space of this system is four-dimensional (it can be identified with the *cotangent bundle* of the manifold, a notion that will be introduced in a later chapter). The system is said to be *conservative* if $\mathbf{f}$ can be written as the gradient of a scalar ($\mathbb{R}$-valued) function,

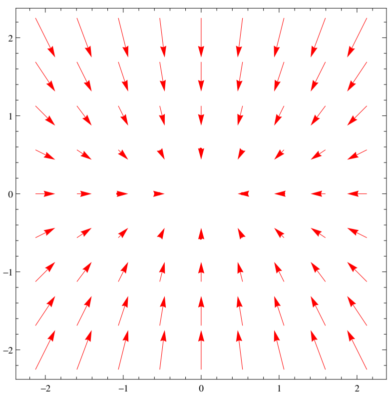
$$\mathbf{f} = -\mathrm{grad} U = -\frac{\partial U}{\partial \mathbf{x}}.$$

As before, we call $U(\mathbf{x})$ the potential energy.

1. Consider $\mathbf{x}$ in the Euclidean plane and $U(\mathbf{x}) = \frac{\omega_1^2}{2}x_1^2 + \frac{\omega_2^2}{2}x_2^2$, $\omega_1,\ \omega_2 > 0$. The total (conserved) energy of the system is $E = \frac{1}{2}(\dot{x_1}^2 + \dot{x_2}^2) + U$. Sketch the vector field $\mathbf{f}$ for $\omega_1 = \omega_2$ and $\omega_1 \neq \omega_2$.



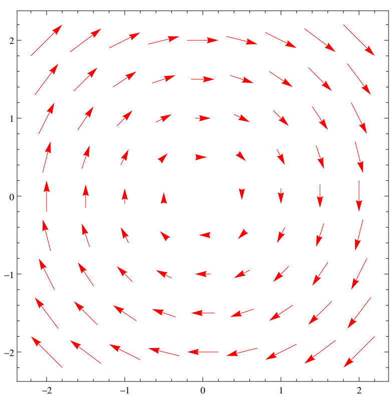
$\omega_2 = \omega_1$



$\omega_2 = 2\omega_1$

2. Find the condition on $\omega_1,\ \omega_2$ such that all orbits are periodic.
   From the general solution we see that the motion in $x_{1,2}$ is periodic with period $\frac{2\pi}{\omega_{1,2}}$. Therefore, the entire orbit is periodic if there exist $n,\ m \in \mathbb{N}$ such that $\frac{n}{\omega_1} = \frac{m}{\omega_2}$, i.e. $\frac{\omega_1}{\omega_2} = \frac{n}{m}$.
3. Find an example for a system which is *not* conservative. Sketch the vector field $\mathbf{f}$.

   For example $\mathbf{f} = (x_2, -x_1)$.



4. Describe (in simple terms) the manifold given by the phase space of an ideal spherical pendulum (i.e. a pendulum whose motion is not restricted to a plane).
   Position space is a two-sphere (S_2) and the velocity is a two-dimensional vector. Therefore, phase space is $S_2 \times \mathbb{R}^2$, a four-dimensional "hypercylinder".