Mécanique II : Série 8

1 Configuration space as differentiable manifold

The configuration space of a system with constraints can be viewed as a differentiable manifold. The dimension of this manifold is called the number of degrees of freedom of the system.

1. The configuration space of a point mass which is constrained to a cylinder of fixed radius R (Série 6, Exercice 1) is, evidently, a cylinder ($\mathbb{R} \times S^1$). Find an atlas for this manifold. Specify the charts explicitly in terms of the cylindrical coordinates and show that they are compatible, i.e. that all transition maps $\varphi'^{-1} \circ \varphi$ are differentiable on their respective domains.
A possible atlas can be constructed in a similar way as an atlas for the circle S^1. For instance, following two mappings from the open strip $]0, 2\pi[\times \mathbb{R} \subset \mathbb{R}^2$ to the cylinder define compatible charts :
$$\varphi_1 : \quad (r, \phi, z) = \quad (R, x_1, y_1) ,$$
$$\varphi_2 : \quad (r, \phi, z) = \begin{cases} (R, x_2 + \pi, y_2) , & \text{if } x_2 < \pi \\ (R, x_2 - \pi, y_2) , & \text{else} \end{cases} ,$$
where (x_1, y_1) and (x_2, y_2) are Cartesian coordinates on the open strip. The transition maps are defined on the preimages of the intersection of the images of φ_1 and φ_2. Since φ_1 excludes $\phi = 0$ and φ_2 excludes $\phi = \pi$, this preimage is in both cases given by $(]0, \pi[\times \mathbb{R}) \cup (]\pi, 2\pi[\times \mathbb{R}) \subset]0, 2\pi[\times \mathbb{R}$. Explicitly, the transition maps are
$$\varphi_2^{-1} \circ \varphi_1 : \quad (x_2, y_2) = \begin{cases} (x_1 + \pi, y_1) , & \text{if } x_1 < \pi \\ (x_1 - \pi, y_1) , & \text{if } x_1 > \pi \end{cases} ,$$
$$\varphi_1^{-1} \circ \varphi_2 : \quad (x_1, y_1) = \begin{cases} (x_2 + \pi, y_2) , & \text{if } x_2 < \pi \\ (x_2 - \pi, y_2) , & \text{if } x_2 > \pi \end{cases} .$$
Evidently, they are differentiable on their entire domain.
In fact, one can cover the cylinder also with a single chart. Consider the following mapping,
$$\varphi : (r, \phi, z) = (R, \phi', \ln r') ,$$
where (r', ϕ') are polar coordinates on $\mathbb{R}^2$. This mapping is a homeomorphism between $U = \mathbb{R}^2 \setminus \{(0,0)\}$ and the whole cylinder and thus defines a chart that covers the entire manifold. The set containing only this chart is automatically an atlas, because no transition functions need to be considered.
2. The configuration space of a point mass which is constrained to a sphere of fixed radius R (Série 6, Exercice 2) is, evidently, a two-sphere (S^2). In Cartesian coordinates, the constraint is given by $x^2+y^2+z^2 = R^2$. A possible atlas of the sphere is given by two charts which use stereographic coordinates, one related to the stereographic projection from the north pole $(x, y, z) = (0, 0, R)$, and one related to the stereographic projection from the south pole $(x, y, z) = (0, 0, -R)$. The corresponding mappings φ_N and φ_S are given by
$$\varphi_N : \quad (x, y, z) = \left(R\frac{2x_N}{1 + x_N^2 + y_N^2}, R\frac{2y_N}{1 + x_N^2 + y_N^2}, R\frac{x_N^2 + y_N^2 - 1}{1 + x_N^2 + y_N^2} \right) ,$$
$$\varphi_S : \quad (x, y, z) = \left(R\frac{2x_S}{1 + x_S^2 + y_S^2}, R\frac{2y_S}{1 + x_S^2 + y_S^2}, -R\frac{x_S^2 + y_S^2 - 1}{1 + x_S^2 + y_S^2} \right) ,$$
where (x_N, y_N) and (x_S, y_S) are Cartesian coordinates in $\mathbb{R}^2$. The *domain* of both mappings (the open sets U_N and U_S on which they are defined) is the entire coordinate space $\mathbb{R}^2$. The *image* of each mapping omits a single point on the sphere : the point from which the stereographic projection is constructed.

Show that the two charts are compatible. To this end, derive explicit expressions for $\varphi_S^{-1} \circ \varphi_N$ and $\varphi_N^{-1} \circ \varphi_S$ to show that they are differentiable on the open subsets V_N and V_S, respectively, which are the corresponding preimages of the intersection of the images of φ_N and φ_S.

From the definition of φ_N and φ_S we immediately see that $x/y = x_N/y_N = x_S/y_S$. From the z-component we get

$$\frac{x_N^2 + y_N^2 - 1}{1 + x_N^2 + y_N^2} = -\frac{x_S^2 + y_S^2 - 1}{1 + x_S^2 + y_S^2},$$

which can be rearranged to

$$\left(x_N^2 + y_N^2\right)\left(x_S^2 + y_S^2\right) = 1\,.$$

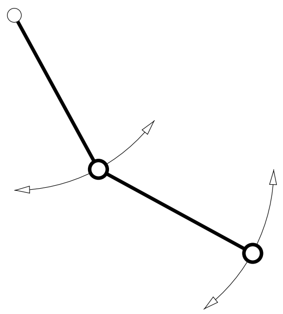
Putting everything together, the transition functions are given explicitly as

$$\varphi_S^{-1} \circ \varphi_N : \quad (x_S, y_S) = \left(\frac{x_N}{(x_N^2 + y_N^2)}, \frac{y_N}{(x_N^2 + y_N^2)}\right),$$

$$\varphi_N^{-1} \circ \varphi_S : \quad (x_N, y_N) = \left(\frac{x_S}{(x_S^2 + y_S^2)}, \frac{y_S}{(x_S^2 + y_S^2)}\right).$$

Evidently, they are differentiable on $V_N = V_S = \mathbb{R}^2 \setminus \{(0,0)\}$.

3. Consider the planar double pendulum (see figure). Its configuration space is the direct product of two circles, $S^1 \times S^1$, which also corresponds to the two-torus T^2. Find an atlas for this manifold. Is there an atlas with only two charts ?



Consider an atlas of the circle which consists of two charts $\varphi : \mathbb{R} \supset U \to S^1$ and $\varphi' : \mathbb{R} \supset U' \to S^1$. (An example is given by the $z = 0$ section of exercise 1.1.) From this atlas of the circle, one can easily construct an atlas for the torus in the following way :

-1cmConsider the mappings $\varphi_1 = (\varphi, \varphi)$, $\varphi_2 = (\varphi, \varphi')$, $\varphi_3 = (\varphi', \varphi)$ and $\varphi_4 = (\varphi', \varphi')$ defined, respectively, on $U_1 = U \times U \subset \mathbb{R}^2$, $U_2 = U \times U' \subset \mathbb{R}^2$, $U_3 = U' \times U \subset \mathbb{R}^2$ and $U_4 = U' \times U' \subset \mathbb{R}^2$. By compatibility of (φ, U) and (φ', U') on S^1, $(\varphi_1, U_1), \ldots, (\varphi_4, U_4)$ are mutually compatible as charts on the torus. Also, each point on the torus is represented in at least one chart. Therefore, $\{(\varphi_1, U_1), \ldots, (\varphi_4, U_4)\}$ is an atlas of T^2.

One can also construct an atlas with only two charts.

Bonus question : give an argument why a single chart cannot be enough.

Assume there *is* a single chart that covers the entire torus. This means there is a homeomorphism between an open subset U of $\mathbb{R}^2$ and the torus T^2. A homeomorphism preserves topological properties like, e.g., compactness. The torus is compact, and therefore U is compact. However, the Bolzano-Weierstrass theorem states that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Therefore, U is closed and bounded. This is in contradiction to the requirement that U is open (U is nonempty), which proves that the assumption ("there *is* a single chart that covers the entire torus") is wrong. More generally, no compact manifold can ever be covered by a single chart.

4. Consider the system of a *closed* chain made up of $n > 2$ identical rigid rods connected by universal joints (joints that allow bending in all directions). Without taking into account the possibility to rotate the rods along their own axes, count the number of degrees of freedom of this system.

A configuration of the system is given by the positions of the n joints. The first joint has three degrees of freedom : moving it around can, for instance, be understood as translating (shifting around) the entire system in three-dimensional space. Each subsequent joint can, in principle, move only in two directions relative to the previous joint, since their separation is fixed by the length of the rods. There are complicated additional geometric constraints which exclude certain combinations of movements, but they do not reduce the number of degrees of freedom. The only exception occurs for the last joint : If all other joints are already positioned, since the chain has to close, the last joint can only move on an arc of a circle. The total number of degrees of freedom therefore is $3 + 2(n - 2) + 1 = 2n$.

5. Consider the previous system with $n = 3$. Which manifold corresponds to its configuration space ?

   With $n = 3$, the chain resembles an equilateral triangle. Its configuration space consists of the three spatial translations combined with any three dimensional rotation. The configuration manifold is therefore diffeomorphic to $\mathbb{R}^3 \times SO(3)$.

2 Embedded manifolds & induced metric

If a manifold $\mathcal{M}$ is embedded into Euclidean space, we can use the Euclidean scalar product to define a symmetric positive-definite bilinear form on the tangent bundle of the manifold, the *induced metric*. It is often convenient to express a metric in terms of the coordinates q_i of a chart of an atlas of $\mathcal{M}$. For instance, its components can easily be read off from the formula for the line element :

$$ds^2 = \sum_{i,j} a_{ij}(\mathbf{q}) dq_i dq_j \, , \; a_{ij} = a_{ji} \, .$$

Given the embedding functions $r_m(\mathbf{q})$, where the r_m are Cartesian coordinates of the Euclidean embedding space, the functions a_{ij} induced by the Euclidean scalar product are computed as

$$a_{ij} = \sum_{m,n} \frac{\partial r_m}{\partial q_i} \frac{\partial r_n}{\partial q_j} \delta_{mn} \, .$$

Consider once again the two-sphere of fixed radius R embedded into $\mathbb{R}^3$. Write the induced line element in terms of the stereographic coordinates.

Straightforward calculation gives

$$ds^2 = \frac{4R^2}{\left(1 + x_N^2 + y_N^2\right)^2} \left[dx_N^2 + dy_N^2\right] = \frac{4R^2}{\left(1 + x_S^2 + y_S^2\right)^2} \left[dx_S^2 + dy_S^2\right] \, .$$

3 Solving problems with constraints

A procedure for solving problems with constraints is given by following recipe.

- Determine the configuration manifold and introduce coordinates q_i. The constraints are implemented here through the embedding.
- Express the kinetic energy in terms of the generalized velocities

$$T = \frac{1}{2} \sum_{i,j} a_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j \, .$$

- Construct the Lagrangean function $L = T - U(\mathbf{q})$ and derive the equations of motion (Euler-Lagrange equations).

Consider once again the cylinder of fixed radius R embedded into $\mathbb{R}^3$. Following the above recipe, derive the equations of motion in each of the charts of the atlas you defined in exercise 1.1, assuming $U \equiv 0$. If possible, solve them.

With the mappings φ_1, φ_2 defined in exercise 1.1, the kinetic energy reads

$$T = \frac{1}{2} R^2 \dot{x}_1^2 + \frac{1}{2} \dot{y}_1^2 = \frac{1}{2} R^2 \dot{x}_2^2 + \frac{1}{2} \dot{y}_2^2 \, .$$

The equations of motion for $U = 0$ are simply $\ddot{x}_1 = \ddot{y}_1 = \ddot{x}_2 = \ddot{y}_2 = 0$. Because of the convenient choice of coordinates, the solutions are orbits of constant coordinate velocities. (One would have been less lucky when one had chosen to use the coordinate map φ instead !)