Mécanique II : Série 10

1 Conserved quantities in time-dependent problems

The damped oscillator is described by the Lagrangean function

$$L(q,\dot q,t) = e^{2\gamma t}\left(\frac{1}{2}\dot q^2 - \frac{1}{2}\omega^2 q^2\right).$$

1. Derive the equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0 \qquad \Rightarrow \qquad \ddot q + 2\gamma \dot q + \omega^2 q = 0\,.$$

In order to study the symmetries of this non-autonomous system, it is useful to consider the *extended configuration space* $\mathcal{M}' \doteq \mathcal{M}\times\mathbb{R}$, where $\mathcal{M}$ is the usual configuration space (for the present problem this is simply $\mathbb{R}$), and the additional $\mathbb{R}$-factor represents the time axis. On the tangent space of this extended configuration manifold, we can define a new Lagrangean function L' as

$$L'\left(q,t,\frac{dq}{d\tau},\frac{dt}{d\tau}\right) \doteq L\left(q,\frac{dq/d\tau}{dt/d\tau},t\right)\frac{dt}{d\tau}\,,$$

where τ is a new time parametrization, and L' is now autonomous with respect to τ. (The original time parameter t is simply treated as an additional generalized coordinate.)

If L' admits a transformation $h^s : \mathcal{M}' \to \mathcal{M}'$, Noether's theorem states that there exists a corresponding first integral $I' : T\mathcal{M}' \to \mathbb{R}$. However, since $\int L' d\tau = \int L dt$, this also gives a first integral $I : T\mathcal{M}\times\mathbb{R} \to \mathbb{R}$ of the original system. The expression for I in local coordinates on $\mathcal{M}'$ in terms of $I'(q,t,dq/d\tau,dt/d\tau)$ is simply

$$I(q,\dot q,t) = I'(q,t,\dot q,1)\,.$$

In the case where the damping goes to zero, $\gamma \to 0$, one finds that L' is independent of t and therefore admits *time translations* $h^s : (q,t) \to (q,t+s)$. The corresponding conserved quantity I is the energy. Energy is no longer conserved in the presence of damping, $\gamma > 0$.

2. Find a transformation h^s with $t \to t+s$ and an appropriate transformation law for q such that h^s is admitted by the Lagrangean function L' in the presence of damping. Obtain the corresponding first integral I.

   The new Lagrangean function

$$L'\left(q,t,\frac{dq}{d\tau},\frac{dt}{d\tau}\right) = e^{2\gamma t}\left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2\left(\frac{dt}{d\tau}\right)^{-2} - \frac{1}{2}\omega^2 q^2\right]\frac{dt}{d\tau}$$

   admits the transformation $h^s : (q,t) \to (e^{-\gamma s}q, t+s)$. This implies the first integral

$$I'\left(q,t,\frac{dq}{d\tau},\frac{dt}{d\tau}\right) = -e^{2\gamma t}\left[\gamma q\frac{dq}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1} + \frac{1}{2}\left(\frac{dq}{d\tau}\right)^2\left(\frac{dt}{d\tau}\right)^{-2} + \frac{1}{2}\omega^2 q^2\right].$$

   The corresponding constant of motion of the original system is therefore

$$I(q,\dot q,t) = -e^{2\gamma t}\left(\gamma q\dot q + \frac{1}{2}\dot q^2 + \frac{1}{2}\omega^2 q^2\right).$$

The general solution of the damped oscillator can be obtained by making the change of variables $q = e^{-\gamma t} r$. Applied to the original equation of motion, this coordinate change gives an *undamped* oscillator equation for the new variable r.

3. Verify this. Determine the general solution for r.

   With $q = e^{-\gamma t} r$ we have $\dot{q} = e^{-\gamma t}\dot{r} - \gamma e^{-\gamma t} r$ and $\ddot{q} = e^{-\gamma t}\ddot{r} - 2\gamma e^{-\gamma t}\dot{r} + \gamma^2 e^{-\gamma t} r$. The equation of motion becomes
   $$\ddot{r} + \left(\omega^2 - \gamma^2\right) r = 0\,,$$
   which is the equation for an undamped oscillator with *shifted frequency* $\omega' \doteq \sqrt{\omega^2 - \gamma^2}$. If $\omega^2 - \gamma^2 < 0$, the shifted frequency becomes imaginary. In this case, the original system is said to be *overdamped*, and the solutions for r are exponentials. The special case $\omega^2 - \gamma^2 = 0$ is referred to as *critical damping*.

   The general solution for r reads
   $$r = \begin{cases} A\sin\omega' t + B\cos\omega' t & \text{if } \omega^2 > \gamma^2 \\ C + Dt & \text{if } \omega^2 = \gamma^2 \\ Ee^{\sqrt{\gamma^2-\omega^2}t} + Fe^{-\sqrt{\gamma^2-\omega^2}t} & \text{if } \omega^2 < \gamma^2 \end{cases}$$
   with A, B, C, D, E, F constants.

4. Using the general solution for r, verify that the first integral I obtained above is indeed a constant of motion.

   Performing the change of variables from q to r, one obtains
   $$I = -\left[\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(\omega^2 - \gamma^2\right) r^2\right]\,,$$
   which resembles the energy integral for the undamped system r. Inserting the general solutions, one finds
   $$I = \begin{cases} -\frac{1}{2}\left(\omega^2 - \gamma^2\right)\left(A^2 + B^2\right) & \text{if } \omega^2 > \gamma^2 \\ -\frac{1}{2}D^2 & \text{if } \omega^2 = \gamma^2 \\ -2\left(\omega^2 - \gamma^2\right) EF & \text{if } \omega^2 < \gamma^2 \end{cases}$$
   which indeed are constants of motion.

2 Solving problems with constraints (revisited)

A procedure for solving problems with constraints is given by following recipe.

- Determine the configuration manifold and introduce coordinates q_i. The constraints are implemented here through the embedding.
- Express the kinetic energy in terms of the generalized velocities
  $$T = \frac{1}{2}\sum_{i,j} a_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j\,.$$
- Construct the Lagrangean function $L = T - U(\mathbf{q})$ and derive the equations of motion (Euler-Lagrange equations).

Consider the ideal spherical pendulum whose configuration manifold is the unit two-sphere S^2, given by the constraint $x^2 + y^2 + z^2 = 1$. In a uniform gravitational field the potential takes the form $U(x, y, z) = cz$. Following the above recipe, one can introduce, for instance, stereographic coordinates on the two-sphere, defined by the mapping $\varphi_N : \mathbb{R}^2 \to S^2$:

$$\varphi_N : (x, y, z) = \left(\frac{2x_N}{x_N^2 + y_N^2 + 1}, \frac{2y_N}{x_N^2 + y_N^2 + 1}, \frac{x_N^2 + y_N^2 - 1}{x_N^2 + y_N^2 + 1}\right)\,,$$

where (x_N, y_N) are Cartesian coordinates in the coordinate space $\mathbb{R}^2$.

1. Construct the Lagrangean function L in stereographic coordinates.

   Straightforward calculation gives

$$L\left(x_N, y_N, \dot{x}_N, \dot{y}_N\right) = 2\frac{\dot{x}_N^2 + \dot{y}_N^2}{\left(x_N^2 + y_N^2 + 1\right)^2} - c\frac{x_N^2 + y_N^2 - 1}{x_N^2 + y_N^2 + 1}\,.$$

2. Find a time-independent transformation h^s of x_N, y_N that leaves $x_N^2 + y_N^2$ invariant and is therefore admitted by L (why ?). What is the corresponding conserved quantity ?

   The only continuous group of transformations (in two dimensions) that preserves distances (and hence $x_N^2 + y_N^2$), maps the origin to itself and is continuously connected to the identity is the rotation group SO(2). Since it also preserves the length of the velocity vector, and L is a function only of $x_N^2 + y_N^2$ and $\dot{x}_N^2 + \dot{y}_N^2$, any SO(2) rotation is admitted by L. Explicitly, the rotations can be parametrized as $h^s : (x_N, y_N) \to (x_N \cos s - y_N \sin s, y_N \cos s + x_N \sin s)$, where s is the rotation angle.

   Using Noether's theorem, one finds following conserved quantity associated to h^s :

$$I = 4\frac{x_N \dot{y}_N - \dot{x}_N y_N}{\left(x_N^2 + y_N^2 + 1\right)^2} = x\dot{y} - \dot{x}y\,,$$

   which is the angular momentum with respect to the z-axis. (The x and y components of the angular momentum vector are not conserved, because the gravitational field $c \neq 0$ breaks rotational invariance around these axes.)