Mécanique II : Série 13

1 Canonical transformations

A one-dimensional harmonic oscillator consisting of a point of unit mass is described by the Hamiltonian

$$H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \,.$$

Canonical equations in which the Hamiltonian function depends only on *one* variable are easy to integrate. For instance, if $H = K(P,t)$, then the canonical equations read

$$\begin{aligned} \dot{Q} &= \frac{\partial K}{\partial P}\,, \\ \dot{P} &= 0\,, \end{aligned}$$

and are solved by

$$\begin{aligned} Q(t) &= Q(0) + \int_0^t \left.\frac{\partial K}{\partial P}\right|_{P(0)} ds\,, \\ P(t) &= P(0)\,. \end{aligned}$$

1. Find a canonical transformation which brings the Hamiltonian function H to the desired form. To this end, try the ansatz $q \to \omega^{-1} f(P)\sin(Q)$, $p \to f(P)\cos(Q)$, which maps $H \to K = f^2(P)/2$. Now construct a generating function $F_1(Q,q)$ which generates this transformation.
   To construct F_1, use $p = \partial F_1/\partial q$:
   $$p = f(P)\cos(Q) = \frac{\omega q}{\sin(Q)}\cos(Q) = \frac{\partial F_1}{\partial q} \;\Rightarrow\; F_1 = \frac{1}{2}q^2\omega\frac{\cos(Q)}{\sin(Q)}\,.$$

2. Using this generating function, express q and p explicitly in terms of Q and P, and determine the new Hamiltonian function K.
   Now use $P = -\partial F_1/\partial Q$:
   $$P = -\frac{\partial F_1}{\partial Q} = \frac{\omega q^2}{2\sin^2(Q)} \;\Rightarrow\; f(P) = \sqrt{2\omega P}\,,$$
   $$q = \sqrt{\frac{2P}{\omega}}\sin(Q)\,, \qquad p = \sqrt{2\omega P}\cos(Q)\,, \qquad K = \omega P\,.$$

3. Since K is a function of P only, P is a constant of motion (related to the energy by solving $K(P) = E$ for P). Determine also the integral solution for Q.
   $$\begin{aligned} \dot{P} &= -\frac{\partial K}{\partial Q} = 0 \;\Rightarrow\; P = \text{const.} = \frac{E}{\omega}\,, \\ \dot{Q} &= \frac{\partial K}{\partial P} = \omega \;\Rightarrow\; Q = \omega t + C\,. \end{aligned}$$

4. Finally, by substituting back into the expressions found in 1.2, give the solutions q, p of the original system.
   $$q = \sqrt{\frac{2E}{\omega^2}}\sin(\omega t + C)\,, \qquad p = \sqrt{2E}\cos(\omega t + C)\,.$$

2 Hamilton–Jacobi

Using the Hamilton–Jacobi equation, one can take advantage of the existence of conserved quantities when solving a dynamical system. As an example, consider the following Hamiltonian function written in polar coordinates :

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + V(r)\,.$$

Since the potential V only depends on the radial coordinate, the angular momentum p_ϕ is conserved. Furthermore, since H does not explicitly depend on time, the energy is also conserved.

One can now try to find a generating function $F_2(P_1, P_2, r, \phi, t)$ which solves the Hamilton–Jacobi equation

$$H\left(p_r = \frac{\partial F_2}{\partial r}, p_\phi = \frac{\partial F_2}{\partial \phi}, r, \phi, t\right) = -\frac{\partial F_2}{\partial t} = K(P_1, P_2, t)\,,$$

where P_1, P_2 are new canonical variables which will correspond to the constants of motion. The function F_2 can be obtained by separation of variables using the following ansatz :

$$F_2(P_1, P_2, r, \phi, t) = R(r, P_1, P_2) + \Phi(\phi, P_2) + T(t, P_1)\,.$$

Inserting this ansatz in the Hamilton–Jacobi equation, one can first obtain the solution for T, which depends on an integration constant P_1. Next, one can solve for Φ which depends on another integration constant P_2. Finally, the solution for R can be constructed, which then depends on both P_1 and P_2. A global constant can always be added to the solutions that will have no effect on the dynamical variables and is therefore omitted from the list of parameters.

1. Following the above instructions, obtain the solutions for T, Φ, R, and finally F_2.

   Inserting the ansatz yields

   $$\frac{1}{2}\left[\left(\frac{\partial R}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\right] + V(r) = -\frac{\partial T}{\partial t}\,.$$

   Since the left-hand-side does not depend on t, while the right-hand-side does not depend on r or ϕ, both sides have to be constant. Therefore,

   $$T = \text{const.} \times t = P_1 t\,.$$

   Furthermore, this solution already implies

   $$K(P_1, P_2, t) = K(P_1) = -P_1\,,$$

   and the integration constant P_1 corresponds to (minus) the conserved energy of the system. Next, separate the variables r and ϕ :

   $$\frac{1}{2}\left(\frac{\partial R}{\partial r}\right)^2 r^2 + V(r)r^2 + P_1 r^2 = -\frac{1}{2}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\,.$$

   Here, the left-hand-side only depends on r, while the right-hand-side only depends on ϕ. Again, both sides have to be constant, such that

   $$\Phi = \text{const.} \times \phi = P_2 \phi\,.$$

   Finally, solving for R yields

   $$\left(\frac{\partial R}{\partial r}\right)^2 = -2P_1 - 2V(r) - \frac{P_2^2}{r^2} \;\Rightarrow\; R = \pm\int dr\sqrt{-2P_1 - 2V(r) - \frac{P_2^2}{r^2}}\,.$$

   Putting everything together, the generating function F_2 reads

   $$F_2(P_1, P_2, r, \phi, t) = P_1 t + P_2 \phi \pm \int dr\sqrt{-2P_1 - 2V(r) - \frac{P_2^2}{r^2}}\,.$$

2. Using the relation $Q_i = \partial F_2/\partial P_i$, construct the new canonical coordinates Q_1 and Q_2.

$$
\begin{aligned}
Q_1 &= \frac{\partial F_2}{P_1} = t \mp \int \frac{dr}{\sqrt{-2P_1 - 2V(r) - \frac{P_2^2}{r^2}}}\,, \\
Q_2 &= \frac{\partial F_2}{P_2} = \phi \mp P_2 \int \frac{dr}{r^2\sqrt{-2P_1 - 2V(r) - \frac{P_2^2}{r^2}}}\,.
\end{aligned}
$$

3. Verify that Q_2 is a constant of motion.

$$
\dot{Q}_2 = \frac{\partial K}{\partial P_2} = 0 \;\Rightarrow\; Q_2 = \text{const.}
$$

Evidently, the evolution in terms of the new variables Q_i, P_i is almost trivial : P_1, P_2 and Q_2 are constant, and Q_1 is a linear function of time. If one can map these solutions back to the original variables, the dynamics are solved.

Consider now the potential $V(r) = -k/r$ and following particular initial conditions :

$$
\begin{aligned}
r(0) &= r_0\,, & \dot{r}(0) &= 0\,, \\
\phi(0) &= \phi_0\,, & \dot{\phi}(0) &= \frac{L}{r_0^2}\,.
\end{aligned}
$$

Further, assume $k > 0$ and $L^2 < 2kr_0$.

4. Write the constants P_1 and P_2 in terms of k and the initial conditions.

$$
\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = p_r \quad \Rightarrow \quad p_r(0) = 0 \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{r^2} \quad \Rightarrow \quad p_\phi(0) = L
\end{aligned}
$$

Evaluating $p_i = \partial F_2/\partial q_i$ at $t = 0$ yields

$$
\left.
\begin{aligned}
0 &= \left.\frac{\partial F_2}{\partial r}\right|_{t=0} = \pm\sqrt{-2P_1 + \frac{2k}{r_0} - \frac{P_2^2}{r_0^2}} \\
L &= \left.\frac{\partial F_2}{\partial \phi}\right|_{t=0} = P_2
\end{aligned}
\right\} \Rightarrow
\begin{cases}
P_1 = \dfrac{k}{r_0} - \dfrac{L^2}{2r_0^2}\,, \\
P_2 = L\,.
\end{cases}
$$

5. Use the expression [1] for Q_2 and the fact that Q_2 is a constant of motion in order to obtain r as a function of ϕ! As the time variable is absent in this relation, it corresponds to a parametric description of the orbit.

1. Following indefinite integral may be useful :

$$
\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \arcsin\left(\frac{bx + 2c}{x\sqrt{b^2 - 4ac}}\right) \qquad \text{if} \quad c < 0 \quad \text{and} \quad b^2 > 4ac
$$

With $V(r) = -k/r$, the integral in Q_2 can be carried out explicitly, giving

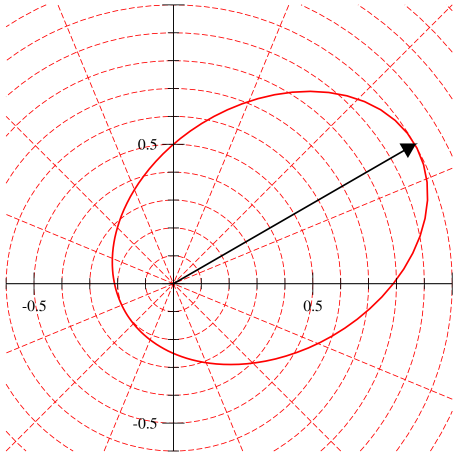
$$Q_2 = \phi \mp \frac{L}{|L|} \arcsin\left(\frac{k - \frac{L^2}{r}}{\left|k - \frac{L^2}{r_0}\right|}\right) .$$

Since Q_2 is a constant of motion, it is given by its value at $t = 0$,

$$Q_2 = \phi_0 \mp \frac{L}{|L|} \arcsin\left(\frac{k - \frac{L^2}{r_0}}{\left|k - \frac{L^2}{r_0}\right|}\right) .$$

Equating the two expressions and solving for r yields

$$r = \frac{L^2}{k - \left(k - \frac{L^2}{r_0}\right)\cos\left(\phi - \phi_0\right)} .$$



Polar plot of $r(\phi)$ (in units of r_0) with initial conditions $\phi_0 = \pi/6$ and $L^2 = kr_0/3$. The initial position on the elliptical orbit is indicated by the black arrow.