Mécanique II : Série 4

1 Systems with one degree of freedom

A system with a single degree of freedom is described by the differential equation

$$\ddot{x} = f(x) = -\frac{dU}{dx}\,,$$

and its conserved total energy is the sum

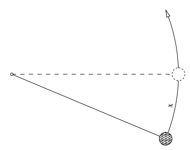
$$E = \frac{1}{2}\dot{x}^2 + U(x)\,.$$

1. Show that the time it takes to reach x_2 from x_1 is equal to
$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2E - 2U(x)}}\,.$$
2. Consider the symmetric potentials $U(x) = \frac{1}{2n}x^{2n}$, $n \in \mathbb{N}$. For $E > 0$, the system performs a periodic motion (oscillation) between the two points $x = \pm a$ where $U(\pm a) = E$. How does the period depend on the amplitude a of the oscillation ?
3. Is there a choice of n for the above potentials where the period is independent of the amplitude ?
4. Sketch the *phase curves* for the cases $n = 1$ and $n = 2$.

An ideal planar pendulum (see figure) is described by the equation

$$\ddot{x} = -\sin(x)\,.$$

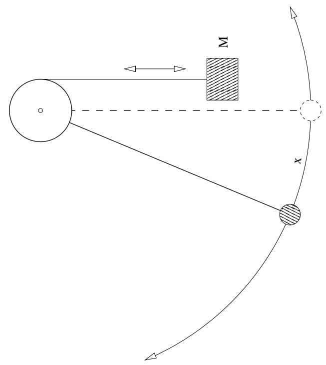
Remark : coordinates which differ by an integer multiple of 2π correspond to the same position of the pendulum. In other words, each coordinate x is mapped to a point on a circle, and each point in the phase plane is mapped to a point on a *phase cylinder*. These maps are called *covers*, and the real line $\mathbb{R}$ and the phase plane are called *universal covering spaces* of the circle and the phase cylinder, respectively.



5. Sketch the potential energy and the phase curves for several choices of total energy. Discuss the three cases $E < U(\pi)$, $E = U(\pi)$, and $E > U(\pi)$. How many distinct phase curves are there for a fixed energy level E ?
6. Sketch the graph of $P(a)$, where P is the period of the periodic motion for given amplitude a. *Hint :* consider the behavior of P and dP/da when a approaches the two critical values $a \to 0$ and $a \to \pi$.

Let a constant torque M be applied to the pendulum. The equation of motion becomes

$$\ddot{x} = -\sin(x) + M\,.$$



7. Sketch the potential energy for the three cases $0 < M < 1$, $M = 1$, and $M > 1$. Identify the *equilibrium points* where $x =$ constant, i.e. $\dot{x} = \ddot{x} = 0$. Which of them are stable?

8. Sketch and discuss various possible phase curves for all three cases mentioned above. Where can one find periodic motion?

2 Phase flow

Let $p(0)$ be a point in the phase plane. The phase curve passing through this point belongs to the solution with initial conditions at $t = 0$ given by $p(0)$. Assuming that the solution can be extended to the entire time axis, its value $p(t)$ is defined at any t and is uniquely determined by $p(0)$. We can introduce a map g^t from the phase space to itself by writing

$$p(t) = g^t p(0)\,.$$

This map is a *diffeomorphism*, i.e. it is bijective, differentiable, and has a differentiable inverse. The set of diffeomorphisms g^t, $t \in \mathbb{R}$ has an abelian group structure : the group operation is the composition $g^{t+s} = g^t \circ g^s = g^s \circ g^t$, the identity element is g^0, and g^{-t} is the inverse of g^t. One says that the transformations g^t form a *one-parameter group of diffeomorphisms* of the phase plane. This group is called the *phase flow*.

1. Show that the system with a single degree of freedom and with potential $U(x) = -x^4$ does not define a phase flow.
2. Show that for systems with a single degree of freedom, to define a phase flow, it is sufficient to have a potential energy which is bounded from below.
3. Sketch the image of a disc in phase space, defined by the condition $x^2 + (\dot{x} - 1)^2 < \frac{1}{4}$, under the action of an element of the phase flow for
   a) the harmonic oscillator $\ddot{x} = -x$, and
   b) the ideal planar pendulum $\ddot{x} = -\sin(x)$.

3 Systems with two degrees of freedom

A system with two degrees of freedom is described by the differential equation

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\,,$$

where $\mathbf{x}$ is a point on a two-dimensional manifold (e.g. the Euclidean plane) and $\mathbf{f}$ is a vector field on this manifold. The phase space of this system is four-dimensional (it can be identified with the *cotangent bundle* of the manifold, a notion that will be introduced in a later chapter). The system is said to be *conservative* if $\mathbf{f}$ can be written as the gradient of a scalar ($\mathbb{R}$-valued) function,

$$\mathbf{f} = -\mathrm{grad} U = -\frac{\partial U}{\partial \mathbf{x}}\,.$$

As before, we call $U(\mathbf{x})$ the potential energy.

1. Consider $\mathbf{x}$ in the Euclidean plane and $U(\mathbf{x}) = \frac{\omega_1^2}{2}x_1^2 + \frac{\omega_2^2}{2}x_2^2$, $\omega_1,\ \omega_2 > 0$. The total (conserved) energy of the system is $E = \frac{1}{2}(\dot{x_1}^2 + \dot{x_2}^2) + U$. Sketch the vector field $\mathbf{f}$ for $\omega_1 = \omega_2$ and $\omega_1 \neq \omega_2$.
2. Find the condition on $\omega_1,\ \omega_2$ such that all orbits are periodic.
3. Find an example for a system which is *not* conservative. Sketch the vector field $\mathbf{f}$.
4. Describe (in simple terms) the manifold given by the phase space of an ideal spherical pendulum (i.e. a pendulum whose motion is not restricted to a plane).