Mécanique II : Série 8

1 Configuration space as differentiable manifold

The configuration space of a system with constraints can be viewed as a differentiable manifold. The dimension of this manifold is called the number of degrees of freedom of the system.

1. The configuration space of a point mass which is constrained to a cylinder of fixed radius R (Série 6, Exercice 1) is, evidently, a cylinder ($\mathbb{R} \times S^1$). Find an atlas for this manifold. Specify the charts explicitly in terms of the cylindrical coordinates and show that they are compatible, i.e. that all transition maps $\varphi'^{-1} \circ \varphi$ are differentiable on their respective domains.

   Remark : We follow the convention of Arnold's book and call φ a map from coordinate space to the manifold and φ^{-1} its inverse. In most of the literature on manifolds, the opposite point of view is usually taken. However, since φ is a homeomorphism, the inverse is as good as the map itself, and it is simply a matter of notation which one you give the symbol φ. Keep this in mind when referring to alternative literature.

2. The configuration space of a point mass which is constrained to a sphere of fixed radius R (Série 6, Exercice 2) is, evidently, a two-sphere (S^2). In Cartesian coordinates, the constraint is given by $x^2+y^2+z^2 = R^2$. A possible atlas of the sphere is given by two charts which use stereographic coordinates, one related to the stereographic projection from the north pole $(x, y, z) = (0, 0, R)$, and one related to the stereographic projection from the south pole $(x, y, z) = (0, 0, -R)$. The corresponding mappings φ_N and φ_S are given by

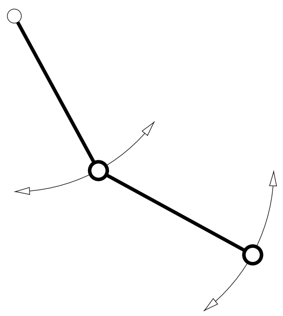
$$\varphi_N : \quad (x,y,z) = \left(R\frac{2x_N}{1+x_N^2+y_N^2}, R\frac{2y_N}{1+x_N^2+y_N^2}, R\frac{x_N^2+y_N^2-1}{1+x_N^2+y_N^2} \right),$$
$$\varphi_S : \quad (x,y,z) = \left(R\frac{2x_S}{1+x_S^2+y_S^2}, R\frac{2y_S}{1+x_S^2+y_S^2}, -R\frac{x_S^2+y_S^2-1}{1+x_S^2+y_S^2} \right),$$

   where (x_N, y_N) and (x_S, y_S) are Cartesian coordinates in $\mathbb{R}^2$. The *domain* of both mappings (the open sets U_N and U_S on which they are defined) is the entire coordinate space $\mathbb{R}^2$. The *image* of each mapping omits a single point on the sphere : the point from which the stereographic projection is constructed.

   Show that the two charts are compatible. To this end, derive explicit expressions for $\varphi_S^{-1} \circ \varphi_N$ and $\varphi_N^{-1} \circ \varphi_S$ to show that they are differentiable on the open subsets V_N and V_S, respectively, which are the corresponding preimages of the intersection of the images of φ_N and φ_S.

3. Consider the planar double pendulum (see figure). Its configuration space is the direct product of two circles, $S^1 \times S^1$, which also corresponds to the two-torus T^2. Find an atlas for this manifold. Is there an atlas with only two charts ?

   Bonus question : give an argument why a single chart cannot be enough.



4. Consider the system of a *closed* chain made up of $n > 2$ identical rigid rods connected by universal joints (joints that allow bending in all directions). Without taking into account the possibility to rotate the rods along their own axes, count the number of degrees of freedom of this system.

5. Consider the previous system with $n = 3$. Which manifold corresponds to its configuration space ?

2 Embedded manifolds & induced metric

If a manifold $\mathcal{M}$ is embedded into Euclidean space, we can use the Euclidean scalar product to define a symmetric positive-definite bilinear form on the tangent bundle of the manifold, the *induced metric*. It is often convenient to express a metric in terms of the coordinates q_i of a chart of an atlas of $\mathcal{M}$. For instance, its components can easily be read off from the formula for the line element :

$$ds^2 = \sum_{i,j} a_{ij}(\mathbf{q}) dq_i dq_j \; , \; a_{ij} = a_{ji} \, .$$

Given the embedding functions $r_m(\mathbf{q})$, where the r_m are Cartesian coordinates of the Euclidean embedding space, the functions a_{ij} induced by the Euclidean scalar product are computed as

$$a_{ij} = \sum_{m,n} \frac{\partial r_m}{\partial q_i} \frac{\partial r_n}{\partial q_j} \delta_{mn} \, .$$

Consider once again the two-sphere of fixed radius R embedded into $\mathbb{R}^3$. Write the induced line element in terms of the stereographic coordinates.

3 Solving problems with constraints

A procedure for solving problems with constraints is given by following recipe.

- Determine the configuration manifold and introduce coordinates q_i. The constraints are implemented here through the embedding.
- Express the kinetic energy in terms of the generalized velocities

$$T = \frac{1}{2} \sum_{i,j} a_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j \, .$$

- Construct the Lagrangean function $L = T - U(\mathbf{q})$ and derive the equations of motion (Euler-Lagrange equations).

Consider once again the cylinder of fixed radius R embedded into $\mathbb{R}^3$. Following the above recipe, derive the equations of motion in each of the charts of the atlas you defined in exercise 1.1, assuming $U \equiv 0$. If possible, solve them.