Mécanique II : Série 10

1 Conserved quantities in time-dependent problems

The damped oscillator is described by the Lagrangean function

$$L(q, \dot{q}, t) = e^{2\gamma t} \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \right) .$$

1. Derive the equation of motion.

In order to study the symmetries of this non-autonomous system, it is useful to consider the *extended configuration space* $\mathcal{M}' \doteq \mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is the usual configuration space (for the present problem this is simply $\mathbb{R}$), and the additional $\mathbb{R}$-factor represents the time axis. On the tangent space of this extended configuration manifold, we can define a new Lagrangean function L' as

$$L' \left(q, t, \frac{dq}{d\tau}, \frac{dt}{d\tau} \right) \doteq L \left(q, \frac{dq/d\tau}{dt/d\tau}, t \right) \frac{dt}{d\tau} ,$$

where τ is a new time parametrization, and L' is now autonomous with respect to τ. (The original time parameter t is simply treated as an additional generalized coordinate.)

If L' admits a transformation $h^s : \mathcal{M}' \to \mathcal{M}'$, Noether's theorem states that there exists a corresponding first integral $I' : T\mathcal{M}' \to \mathbb{R}$. However, since $\int L' d\tau = \int L dt$, this also gives a first integral $I : T\mathcal{M} \times \mathbb{R} \to \mathbb{R}$ of the original system. The expression for I in local coordinates on $\mathcal{M}'$ in terms of $I'(q, t, dq/d\tau, dt/d\tau)$ is simply

$$I(q, \dot{q}, t) = I'(q, t, \dot{q}, 1) .$$

In the case where the damping goes to zero, $\gamma \to 0$, one finds that L' is independent of t and therefore admits *time translations* $h^s : (q, t) \to (q, t + s)$. The corresponding conserved quantity I is the energy. Energy is no longer conserved in the presence of damping, $\gamma > 0$.

2. Find a transformation h^s with $t \to t + s$ and an appropriate transformation law for q such that h^s is admitted by the Lagrangean function L' in the presence of damping. Obtain the corresponding first integral I.

The general solution of the damped oscillator can be obtained by making the change of variables $q = e^{-\gamma t} r$. Applied to the original equation of motion, this coordinate change gives an *undamped* oscillator equation for the new variable r.

3. Verify this. Determine the general solution for r.
4. Using the general solution for r, verify that the first integral I obtained above is indeed a constant of motion.

2 Solving problems with constraints (revisited)

A procedure for solving problems with constraints is given by following recipe.

- Determine the configuration manifold and introduce coordinates q_i. The constraints are implemented here through the embedding.
- Express the kinetic energy in terms of the generalized velocities

$$T = \frac{1}{2} \sum_{i,j} a_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j \,.$$

- Construct the Lagrangean function $L = T - U(\mathbf{q})$ and derive the equations of motion (Euler-Lagrange equations).

Consider the ideal spherical pendulum whose configuration manifold is the unit two-sphere S^2, given by the constraint $x^2 + y^2 + z^2 = 1$. In a uniform gravitational field the potential takes the form $U(x, y, z) = cz$. Following the above recipe, one can introduce, for instance, stereographic coordinates on the two-sphere, defined by the mapping $\varphi_N : \mathbb{R}^2 \to S^2$:

$$\varphi_N : (x, y, z) = \left(\frac{2x_N}{x_N^2 + y_N^2 + 1}, \frac{2y_N}{x_N^2 + y_N^2 + 1}, \frac{x_N^2 + y_N^2 - 1}{x_N^2 + y_N^2 + 1} \right) ,$$

where (x_N, y_N) are Cartesian coordinates in the coordinate space $\mathbb{R}^2$.

1. Construct the Lagrangean function L in stereographic coordinates.
2. Find a time-independent transformation h^s of x_N, y_N that leaves $x_N^2 + y_N^2$ invariant and is therefore admitted by L (why ?). What is the corresponding conserved quantity ?