Quantum Mechanics I, Correction Sheet 11, Spring 2013

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May 29, 2013 (Ecole de Physique, Auditoire Stuckelberg)

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I. FERMI'S GOLDEN RULE

Consider a system described by a time-independent Hamiltonian $\hat{H}_0$, and let the system be in a particular state $|E_i\rangle$, which is an eigenstate of the Hamiltonian, $\hat{H}_0 |E_i\rangle = E_i |E_i\rangle$, where E_i is the energy of the state.

At time $t = 0$ a small time-dependent perturbation of the form

$$\hat{W}(t) = \hat{V} e^{-i\omega t} \tag{1}$$

is switched on for a period of time T, then it is switched off. We ask ourselves: What is the average probability per unit of time of the transition from $|E_i\rangle$ to some other state $|E_f\rangle$? In other words, what is the transition rate between the states?

The amplitude of the transition between the initial and final states, $|E_i\rangle$ and $|E_f\rangle$, respectively, is defined using the evolution operator

$$A_{fi}(T) = \left\langle E_f \middle| \hat{U}(T) \middle| E_i \right\rangle,$$

where $\hat{U}(t)$ is the evolution operator introduced in previous sheets.

Important note: The interpretation of the formula for the amplitude was not entirely correct as given in the tutorial. The main source of confusion is that states $|E_i\rangle$ and $|E_f\rangle$ must be treated here as just some basis states (we can do that because they are, in fact, eigenstates of another Hamiltonian, namely $\hat{H}_0$).

Being construed as a basis state, the initial state can be evolved (or propagated) using the evolution operator, and the amplitude A_{fi} can then be interpreted as the quantity showing what fraction of the final state $|E_f\rangle$ is contained in the evolved state $\hat{U}(t) |E_i\rangle$.

The quantity that we want to evaluate is

$$w_{fi} = \frac{1}{T} P_{fi}(T) = \frac{1}{T} |A_{fi}(T)|^2 . \tag{2}$$

To find the evolution operator we resort to the interaction picture introduced in the previous problem sheet. Recall that the evolution operator in the interaction picture satisfies the integral equation,

$$\hat{U}_I(t) = \hat{I} - \frac{i}{\hbar}\int_0^t dt' \hat{W}_I(t')\hat{U}_I(t').$$

When $\hat{W}(t)$ is a small perturbation, one can solve the equation by iterations,

$$\begin{aligned}
\hat{U}_I^{(0)}(t) =& \hat{I}, \\
\hat{U}_I^{(1)}(t) =& \hat{I} - \frac{i}{\hbar}\int_0^t dt' \hat{W}_I(t')\hat{U}_I^{(0)}(t'), \\
\hat{U}_I^{(2)}(t) =& etc...
\end{aligned}$$

1. Show that the transition amplitude for states with different energies $E_f \neq E_i$ can be written to first order as

$$A_{fi}^{(1)}(t) = -\frac{i}{\hbar}e^{-\frac{i}{\hbar}E_f t}\int_0^t dt' e^{\frac{i}{\hbar}(E_f - E_i)t'} \left\langle E_f \middle| \hat{W}(t') \middle| E_i \right\rangle.$$

[*Hint:* Do not forget to transform back from the interaction to the Schrödinger picture.]

Recalling that $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}_0 t}\hat{U}_I(t)$, we have

$$\begin{aligned}
A_{fi}^{(1)}(t) =& \left\langle E_f \middle| e^{-\frac{i}{\hbar}\hat{H}_0 t}\hat{U}_I^{(1)}(t) \middle| E_i \right\rangle \\
=& e^{-\frac{i}{\hbar}E_f t}\left[\underbrace{\langle E_f | E_i \rangle}_{=0 \text{ for } E_f \neq E_i} - \frac{i}{\hbar}\int_0^t dt' \left\langle E_f \middle| \hat{W}_I(t') \middle| E_i \right\rangle \right] \\
=& \frac{i}{\hbar}e^{-\frac{i}{\hbar}E_f t}\int_0^t dt' \left\langle E_f \middle| e^{\frac{i}{\hbar}\hat{H}_0 t'}\hat{W}(t')e^{-\frac{i}{\hbar}\hat{H}_0 t'} \middle| E_i \right\rangle \\
=& -\frac{i}{\hbar}e^{-\frac{i}{\hbar}E_f t}\int_0^t dt' e^{\frac{i}{\hbar}(E_f - E_i)t'} \left\langle E_f \middle| \hat{W}(t') \middle| E_i \right\rangle.
\end{aligned}$$

2. It is convenient to introduce the transition frequency $\omega_{fi} := (E_f - E_i)/\hbar$. Use the obtained amplitude and Eq. (2) to demonstrate that for a perturbation given by Eq. (1) acting during the period of time T (starting from $t = 0$), the transition rate reads as follows:

$$w_{fi} = \frac{1}{T}\frac{1}{\hbar^2}\left| \left\langle E_f \middle| \hat{V} \middle| E_i \right\rangle \right|^2 \left(\int_0^T dt' \cos(\omega_{fi} - \omega)t' \right)^2.$$

By substituting $\hat{W}(t) = \hat{V}e^{-i\omega t}$ to the expression for the amplitude, we get for the transition rate,

$$\begin{aligned}
w_{fi} =& \frac{1}{T}\left| A_{fi}^{(1)}(T) \right|^2 = \frac{1}{T}\left[\frac{1}{\hbar^2}\left| \int_0^T dt' e^{i\omega_{fi}t'} \left\langle E_f \middle| \hat{V}e^{-i\omega t'} \middle| E_i \right\rangle \right|^2 \right] \\
=& \frac{1}{T}\frac{1}{\hbar^2}\left| \left\langle E_f \middle| \hat{V} \middle| E_i \right\rangle \right|^2 \left| \int_0^T dt' e^{i(\omega_{fi} - \omega)t'} \right|^2 \\
=& \frac{1}{T}\frac{1}{\hbar^2}\left| \left\langle E_f \middle| \hat{V} \middle| E_i \right\rangle \right|^2 \left(\int_0^T dt' \cos\left(\omega_{fi} - \omega\right)t' \right)^2
\end{aligned}$$

3. (*) Evaluate the integral and prove that

$$\frac{1}{T}\left(\int_0^T dt' \cos(\omega_{fi}-\omega)t'\right)^2 \to \pi\delta(\omega_{fi}-\omega), \quad \text{a } T\to\infty.$$

[*Hint:* You will have to use the following improper integral:

$$\int_{\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \pi$$

]

The evaluation of the integral is straightforward,

$$\left(\int_0^T dt' \cos(\omega_{fi}-\omega)t'\right)^2 = \frac{\sin^2(\omega_{fi}-\omega)T}{(\omega_{fi}-\omega)^2}.$$

Let us now prove that the function

$$f_T(x) = \frac{\sin^2 xT}{Tx^2},$$

tends to $\pi\delta(x)$ as $T\to\infty$.

First, we notice that as the sine function is bounded by 1, for any finite value of $x\neq 0$, $f_T(x)\to 0$ as $T\to\infty$. Consider now $x \ll 1/T$, for which

$$f_T(x) \approx \frac{x^2T^2}{Tx^2} = T.$$

We see that $f_T(0)\to\infty$ $(T\to\infty)$. The integral of the function,

$$\int_{-\infty}^{\infty} dx f_T(x) = \int_{-\infty}^{\infty} d(Tx)\frac{\sin^2(xT)}{(xT)^2} = \pi,$$

according to the hint.

This proves that $f_T(x)\to\pi\delta(x)$ as $T\to\infty$.

The transition rate can then be written as

$$\begin{aligned} w_{fi} =& \frac{\pi}{\hbar^2}\left|\left\langle E_f\middle|\hat{V}\middle|E_i\right\rangle\right|^2 \delta(\omega_{fi}-\omega) \\ =& \frac{\pi}{\hbar}\left|\left\langle E_f\middle|\hat{V}\middle|E_i\right\rangle\right|^2 \delta(E_f-E_i-\hbar\omega). \end{aligned}$$

The resulting expression for the transition rate,

$$w_{fi}(\omega) = \frac{\pi}{\hbar^2}\left|\left\langle E_f\middle|\hat{V}\middle|E_i\right\rangle\right|^2 \delta(\omega_{fi}-\omega) \equiv \frac{\pi}{\hbar}\left|\left\langle E_f\middle|\hat{V}\middle|E_i\right\rangle\right|^2 \delta(E_f-E_i-\hbar\omega),$$

is a version of a formula known as "the Fermi's golden rule" which was first derived by P. Dirac and later referred to as the "Golden Rule N. 2" by E. Fermi.

II. NEUTRINO OSCILLATIONS

Neutrinos are very light particle that have three flavors: electron ν_e, muon ν_μ, tau ν_τ. Consider for simplicity only two flavors, ν_e and ν_μ. In numerous experiments it has been established that states with a definite flavor do not coincide with states with a definite mass m_i. This implies that the states are related to each other by a unitary matrix U which can be written as follows,

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = U \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix},$$

where $|\nu_e\rangle$, $|\nu_\mu\rangle$ are flavor states, $|\nu_1\rangle$, $|\nu_2\rangle$ mass states.

In vacuum the propagation of particles is described by a free Hamiltonian, which amounts to simple evolution of mass eigenstates,

$$|\nu_i(t)\rangle = e^{-\frac{i}{\hbar}(E_i t - p_i x)} |\nu_i(t=0)\rangle \, ,$$

where the relativistic energy is $E_i = \sqrt{p_i^2 c^2 + m_i^2 c^4}$ and m_i is the mass of the eigenstate.
This evolution equation can be written in terms of (time-dependent) phases $\phi_i = (E_i t - p_i x)/\hbar$,

$$\begin{pmatrix} |\nu_1(t)\rangle \\ |\nu_2(t)\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{-i\phi_2} \end{pmatrix} \begin{pmatrix} |\nu_1(0)\rangle \\ |\nu_2(0)\rangle \end{pmatrix}.$$

1. Neutrinos are always produced in a state with a definite flavor. Consder an electron neutrino ν_e produced at $t=0$. Find the equation for the evolution of the flavor states $|\nu_e(t)\rangle$, $|\nu_\mu(t)\rangle$.

   [*Hint:* Find the mass states at $t = 0$, evolve them and transform back to flavor states.]

   According to the hint we first evaluate the wave function in the mass state basis,

$$\begin{pmatrix} |\nu_1(t=0)\rangle \\ |\nu_2(t=0)\rangle \end{pmatrix} = U^\dagger \begin{pmatrix} |\nu_e(t=0)\rangle \\ |0\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \, |\nu_e(t=0)\rangle \\ \sin\theta \, |\nu_e(t=0)\rangle \end{pmatrix}.$$

   Evolving them and transforming back we finally get the sought-after eqution,

$$\begin{aligned} \begin{pmatrix} |\nu_e(t)\rangle \\ |\nu_\mu(t)\rangle \end{pmatrix} =& U \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{-i\phi_2} \end{pmatrix} \begin{pmatrix} |\nu_1(t=0)\rangle \\ |\nu_2(t=0)\rangle \end{pmatrix} \\ =& U \begin{pmatrix} e^{-i\phi_1}\cos\theta \, |\nu_e(0)\rangle \\ e^{-i\phi_2}\sin\theta \, |\nu_e(0)\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\phi_1}\cos^2\theta + e^{-i\phi_2}\sin^2\theta \, |\nu_e(0)\rangle \\ \cos\theta\sin\theta(e^{-i\phi_2} - e^{-i\phi_1}) \, |\nu_e(0)\rangle \end{pmatrix}. \end{aligned}$$

2. Evaluate the probability of finding a neutrino in a state $|\nu_\mu(t)\rangle$, i.e. $P(\nu_e \to \nu_\mu)(t) = |\langle\nu_\mu(t)|\nu_e(t=0)\rangle|^2$.

Taking only the ν_μ-part of the equation (and bearing in might that we need the complex conjugate of the expression to get $\langle \nu_\mu(t)|$), we obtain

$$\begin{aligned} P(\nu_e \to \nu_\mu)(t) =& \left| \cos\theta \sin\theta (e^{i\phi_2} - e^{i\phi_1}) \langle \nu_e(0)|\nu_e(0)\rangle \right|^2 \\ =& \cos^2\theta \sin^2\theta (e^{i\phi_2} - e^{i\phi_1})(e^{-i\phi_2} - e^{-i\phi_1}) \\ =& \cos^2\theta \sin^2\theta [2 - 2\cos(\phi_2 - \phi_1)] = \sin^2(2\theta) \sin^2\left(\frac{\phi_2 - \phi_1}{2}\right). \end{aligned}$$

3. Assume the the neutrino has traveled a distance L from the place where it was produced to the detector. The probability of a neutrino transmuting from ν_e to flavor ν_μ can be written as

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta) \sin^2\left(\frac{\phi_2 - \phi_1}{2}\right),$$

where the phase difference depends on the travel time and hence on L.

Since neutrinos are very light particles they can practically always be treated in the ultrarelativistic limit $mc^2 \ll pc$. Expressing the momentum in terms of the energy $p_i c = \sqrt{E_i - m_i^2 c^4}$ and expanding to first order in m_i^2/E_i evaluate the phase difference $\Delta\phi_{21} = \phi_2 - \phi_1$ and express it in terms of the distance L. Also, assume in calculations that the energies of different mass states are equal, $E_1 = E_1 = E$ (this assumption can be relaxed but makes the calculations much more messy).

[*Hint:* The velocity of neutrinos is practically indistinguishable from the light-speed c.]

In the ultrarelativistic limit

$$p_i c = \sqrt{E_i - m_i^2 c^4} \approx Ei\left(1 - \frac{m_i^2 c^4}{2E_i}\right) = E_i - \frac{m_i^2 c^4}{2E_i}.$$

The phase difference is, then,

$$\begin{aligned} \Delta\phi_{21} =& \frac{1}{\hbar}\left[(E_2 - E_1)t - (p_2 - p_1)L\right] \\ =& \frac{1}{\hbar}\left[(E_2 - E_1)\frac{L}{c} - \frac{1}{c}(E_2 - E_1)L - \frac{1}{c}\left(\frac{m_2^2 c^4}{2E_2} - \frac{m_1^2 c^4}{2E_1}\right)L\right] \\ =& \frac{c^3}{\hbar}\left(\frac{m_2^2}{2E_2} - \frac{m_1^2}{2E_1}\right)L = \frac{c^3}{\hbar}\frac{\Delta m_{21}^2}{2E}L, \end{aligned}$$

where in the last line we have used the assumption $E_1 = E_2 = E$ and introduced $\Delta m_{21}^2 = m_2^2 - m_1^2$.

4. If one puts all the units together and assumes GeV as units of energy E, kilometers as units of distance L, and eV as units of neutrino mass, then one obtains the following formula for the oscillation probability

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta) \sin^2\left(1.27\Delta m_{21}^2 \frac{L}{E}\right),$$

where $\Delta m_{21}^2 = m_2^2 - m_1^2$.

Take $\sin^2(2\theta) = 0.8$, $E = 1$ GeV, $\Delta m_{21}^2 = 3 \times 10^{-3}$ (eV)^2, $L = 400$ km, and estimate what fraction of ν_e-neutrinos with this energy reaches the detector.

Inserting the given numbers into the formula we get for the probability,

$$P(\nu_e \to \nu_\mu) = 0.8 \cdot \sin^2\left(1.27 \cdot 3 \times 10^{-3} \cdot 400\right) \approx 0.798.$$

The fraction of ν_e-neutrinos is then

$$\frac{N_{\nu_e}(T)}{N_{\nu_e}(0)} = 1 - P(\nu_e \to \nu_\mu) \approx 0.202.$$

As one can see only 20% of neutrinos of the initial flavor, ν_e, reach the detector. The rest of them have got transformed into ν_μ-neutrinos.