Quantum Mechanics I, Sheet 2, Spring 2013

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I. DE BROGLIE WAVELENGTHS

On sufficiently small length scales, particles exhibit wave-like properties and their wavelength named after de Broglie is given by $\lambda = h/p$ where p is the momentum of the particle. The momentum is related to the kinetic energy via $E_{kin} = E - E_{p=0} = \sqrt{(p_e c)^2 + (m_e c^2)^2} - m_e c^2$, where m_e is the mass and p_e the momentum of the electron. Calculate λ for

(i) an electron whose kinetic energy is 500 KeV (use the relativistic expression for the energy of the electron),

(ii) an electron whose kinetic energy is 30 eV (use the non-relativistic expression for the energy of the electron).

II. A SIMPLE WAVE FUNCTION

We consider an exponentially decaying wave function, defined on a domain $0 \leq r < \infty$,

$$\psi(r) = Ne^{-\frac{r}{2a}}$$

where N is a normalization factor and a is a known real parameter. Such type of wave function describes for example a particle penetrating into a strong potential well.

a. Calculate the factor N.

b. Calculate the expectation values $\langle r \rangle$ and $\langle r^2 \rangle$ of this state.

c. Calculate the probability of finding the particle in the region $r > \langle r \rangle$.

d. Calculate the momentum-space representation of the symmetrized wave function $\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dr e^{-ipr/\hbar} \psi(|r|)$.

[For this exercise we suggest that you explicitly perform the partial integrations. But you can use the following integral to check your results:

$$\int_{\bar{x}}^{+\infty} \mathrm{d}x x^n e^{-x} = \left[(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} \int_{\bar{x}}^{+\infty} \mathrm{d}x e^{-\alpha x} \right]_{\alpha=1},$$

which, in the case $\bar{x} = 0$, reduces to

$$\int_0^{+\infty} \mathrm{d}x x^n e^{-x} = \Gamma(n+1) = n!,$$

where Γ is the gamma function. $\Big]$

III. SCHROEDINGER EQUATION IN MOMENTUM SPACE

A particle of mass m is subjected to a one-dimensional potential $V(x)$ such that the Fourier transform of the wave function $\varphi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx e^{-ipx/\hbar} \psi(x,t)$ satisfies a momentum-space Schroedinger equation given by

$$\left(\frac{p^2}{2m} - \frac{\kappa\hbar^2}{2} \frac{\partial^2}{\partial p^2} \right) \varphi(p,t) = i\hbar \frac{\partial \varphi(p,t)}{\partial t}, \tag{1}$$

where κ is some real constant with suitable physical dimensions. By using Fourier transform, extract the potential $V(x)$ and the corresponding force. What is the physical meaning of κ?