Quantum Mechanics I, Sheet 3, Spring 2013

Responsible for this sheet: P. Guarato (pietro.guarato@unige.ch), office 203, Sciences I

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Prof. D. van der Marel (dirk.vandermarel@unige.ch)

Tutorials: P. Barmettler (peter.barmettler@unige.ch), P. Guarato (pietro.guarato@unige.ch)

I. THE GAUSSIAN WAVE PACKET

Consider an initial gaussian wave packet

$$\varphi(t=0,p) = (\pi\sigma^2\hbar^2)^{-1/4}\exp\left(-\frac{(p-p_0)^2}{2\sigma^2\hbar^2}\right).$$

The equation of motion is given by the Schrödinger equation of a free particle

$$i\hbar\frac{\partial}{\partial t}\varphi(t,p) = \frac{p^2}{2m}\varphi(t,p)\,.$$

(a) Find the Fourier transform $\psi(0,x)$ of $\varphi(0,p)$, at $t=0$.

(b) For $t=0$, show that $\Delta x\Delta p = \hbar/2$.

(c) Show that the spatial width of the wave packet at time t is given by

$$(\Delta x(t))^2 = \frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{t^2\sigma^2\hbar^2}{m^2}\right).$$

[Hint: to calculate some integrals in this exercise, it could sometimes be useful to try to manipulate the standard Gaussian integral, $\int_{-\infty}^{+\infty} e^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}$. For example, $\int_{-\infty}^{+\infty} x^2 e^{-ax^2}\mathrm{d}x = -\frac{\partial}{\partial a}\int_{-\infty}^{+\infty} e^{-ax^2}\mathrm{d}x = -\frac{\partial}{\partial a}\sqrt{\frac{\pi}{a}} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$.]

II. PHYSICAL MEASUREMENTS

(a) Consider the observable $\hat{A}$ of a physical quantity A, and its normalized (and orthogonal) eigenfunctions $\psi_n(\mathbf{r})$ associated to the eigenvalues a_n ($n=1,2$). Calculate the variance $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$ when the wave function of the system is: (i) $\psi_1(\mathbf{r})$ (ii) $\psi(\mathbf{r}) = c_1\psi_1(\mathbf{r}) + c_2 e^{i\phi}\psi_2(\mathbf{r})$, where c_1, c_2 and ϕ are real constants, and $\psi(\mathbf{r})$ is normalized. [Hint: two eigenfunctions $\psi_n(\mathbf{r})$ and $\psi_m(\mathbf{r})$ are orthogonal when $\int \psi_n^*(\mathbf{r})\psi_m(\mathbf{r})\mathrm{d}^3r = \delta_{n,m}$.]

(b) Consider a particle in a one-dimensional system. At the time $t=0$, the state of the system is described by the wave function $\psi(x,0)$, and one measures the position x of the particle immediately after $t=0$. This process is repeated 10 times, and one finds the following results (in nm) : 550, 478, 539, 498, 541, 497, 455, 496, 500, 479.

(i) Calculate the expectation value $\langle x \rangle$ and the variance $(\Delta x)^2$ of the position. Since the probability law $|\psi(x,0)|^2$ is unknown, we will use the following formulas:

$$\langle x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad , \qquad (\Delta x)^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \langle x \rangle)^2 \tag{1}$$

(ii) One repeats the experiment, but immediately after every measurement of the position, one performs a new measurement of the position. What are the results for the expectation value and the variance after this series of measurements?

III. PARTICLE IN A ONE-DIMENSIONAL POTENTIAL

A particle of mass m moves in one dimension under the influence of a potential $V(x)$. Suppose it is in an energy eigenstate $\psi(x) = (\gamma^2/\pi)^{1/4} e^{-\gamma^2 x^2/2}$ with energy $E = \hbar^2\gamma^2/(2m)$.
(a) Find the mean position of the particle, $\langle x \rangle$.
(b) Find the mean momentum of the particle, $\langle p \rangle$.
(c) Find V(x) by using the time-independent Schroedinger equation.