Quantum Mechanics I, Sheet 4, Spring 2013

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I. TIME EVOLUTION OF A SYSTEM

Let us consider two eigenfunctions, $\psi_1(x)$ and $\psi_2(x)$, of a given Hamiltonian whose eigenvalues are E_1 and E_2. We define two states which are symmetric and antisymmetric superpositions of $\psi_1(x)$ and $\psi_2(x)$: $\psi_s(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$ and $\psi_a(x) = \frac{1}{\sqrt{2}}(\psi_1(x) - \psi_2(x))$.

(a) At time $t = 0$, the system is in the state described by the wave function $\psi_1(x)$. What is the probability of finding the system at time t in the state described by the wave function $\psi_a(x)$?

(b) At time $t = 0$, the system is in the state described by the wave function $\psi_s(x)$. What is the probability of finding the system at time t in the state described by the wave function $\psi_a(x)$?

(c) At time $t = 0$, the system is in the state described by the wave function $\psi_s(x)$. At time t_1, one performs a measurement of the energy of the system. What are the possible results? What is the probability that the result is E_1?
If one measures the energy E_1, what is the probability of finding the system in the state described by the wave function ψ_2 after a time $t_2 > t_1$?

II. A SIMPLE WAVE FUNCTION AND THE ASSOCIATED PROBABILITY CURRENT

Consider a normalized wave function $\psi(x)$. Assume that the system is in the state described by the wave function

$$\Psi(x) = C_1\psi(x) + C_2\psi^*(x),$$

where C_1 and C_2 are two known complex numbers.
(a) Write down the condition for the normalization of Ψ in terms of the complex integral $\int_{-\infty}^{+\infty} \psi^2(x)\mathrm{d}x = D$, assumed to be known.
(b) Obtain an expression for the probability current density $J(x)$ for the state $\Psi(x)$. Use the polar relation $\psi(x) = f(x)e^{i\theta(x)}$, where $f(x)$ and $\theta(x)$ are two real functions.

(c) Calculate the expectation value $\langle p \rangle$ of the momentum in the state $\Psi(x)$ and show that

$$\langle p \rangle = m \int_{-\infty}^{+\infty} J(x) \mathrm{d}x.$$

To obtain this result, one has to assume that the function $f(x)$ vanishes at infinity. Show that both the probability current and the momentum vanish if $|C_1| = |C_2|$.

III. COMMUTATION OF OBSERVABLES

(a) $\hat{A}$, $\hat{B}$ and $\hat{C}$ are some observables. Show that : $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$.

(b) The operator of the angular momentum is defined by $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. Calculate the following commutators:

(i) $[\hat{L}_k, \hat{L}_j]$, where $k, j = x, y, z$

(ii) $[\hat{L}_k, \hat{r}_j]$, where $k, j = x, y, z$

(iii) $[\hat{L}_k, \hat{p}_j]$, where $k, j = x, y, z$

(iv) $[\hat{p}^2, \hat{L}_j]$, where $k, j = x, y, z$ [Hint: use (a)]