Quantum Mechanics I, Sheet 5, Spring 2013

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I. THE ONE-DIMENSIONAL QUANTUM HARMONIC OSCILLATOR AND THE HERMITE FUNCTIONS

As explained in Section 4.2 of Basdevant and Dalibard's textbook, the normalized eigenfunctions $\psi_n(x)$ of the one-dimensional quantum harmonic oscillator are

$$\psi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n! a}} e^{-x^2/(2a^2)} H_n(x/a), \tag{1}$$

where $a \equiv \sqrt{\hbar/(m\omega)}$ and $H_n(x/a)$ are called "Hermite functions" and are defined by

$$H_n(y) = (-1)^n e^{y^2} \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(e^{-y^2}\right).$$

The eigenfunctions $\psi_n(x)$ are real and orthogonal, that is,

$$\int \psi_n^*(x)\psi_m(x)\mathrm{d}x = \delta_{n,m}.$$

Using the definition (1), show that these functions satisfy the following recursion relations:

$$x\sqrt{2}\psi_n(x) = a\sqrt{n+1}\psi_{n+1}(x) + a\sqrt{n}\psi_{n-1}(x),$$

$$a\sqrt{2}\frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x) = \sqrt{n}\psi_{n-1}(x) - \sqrt{n+1}\psi_{n+1}(x).$$

II. ONE-DIMENSIONAL HARMONIC OSCILLATOR

At time $t = 0$ a particle in the one-dimensional potential $V(x) = m\omega^2 x^2/2$ is described by the wave function

$$\psi(x,0) = A\sum_n (1/\sqrt{2})^n \psi_n(x),$$

where $\psi_n(x)$ are eigenstates of the energy with eigenvalues $E_n = (n+1/2)\hbar\omega$. You are given that $\int \psi_n^*(x)\psi_{n'}(x)\mathrm{d}x = \delta_{n,n'}$.

(a) Find the value of the normalisation constant A.

(b) Write an expression for $\psi(x,t)$ for $t > 0$ in terms of the eigenstates $\psi_n(x)$.

(c) Show that $|\psi(x,t)|^2$ is a periodic function of time and indicate the longest period τ.

(d) Find the expectation value of the energy at $t = 0$. [Hint: use the fact that $\sum_{n=0}^{\infty} \frac{1}{x^n} = \frac{x}{x-1}$ and hence, by differentiation, $\sum_{n=0}^{\infty} \frac{-n}{x^{n+1}} = \frac{-1}{(x-1)^2}$, in the case $x = 2$.]

III. ONE-DIMENSIONAL WAVE FUNCTION

Consider the one-dimensional wave function

$$\psi(x) = A\left(\frac{x}{x_0}\right)^n e^{-x/x_0},$$

where A, n and x_0 are constants. Using the time-independent Schroedinger equation, find the potential $V(x)$ and energy E for which the wave function is an eigenfunction. (Assume that as $x \to \infty$, $V(x) \to 0$).

IV. <u>OPTIONAL</u>: NONDEGENERATE SOLUTION OF THE ONE-DIMENSIONAL SCHROEDINGER EQUATION

Consider the one-dimensional time-independent Schroedinger equation for some arbitrary potential $V(x)$. Prove that if a solution $\psi(x)$ has the property that $\psi(x) \to 0$ as $x \to \pm\infty$, then the solution must be nondegenerate (that is, unique up to a constant phase). [Hint: show that the contrary assumption leads to a contradiction.][<u>This exercise is optional</u>.]