Quantum Mechanics I, Sheet 6, Spring 2013

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April 10, 2013 (EP, Auditoire Stuckelberg)

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I. HERMITIAN AND UNITARY OPERATORS

Let $\hat{A}$ be a linear operator. The conjugate operator $\hat{A}^\dagger$ is defined by the relation

$$\langle\phi|\hat{A}|\psi\rangle = \Big(\langle\psi|\hat{A}^\dagger|\phi\rangle\Big)^* ,$$

which holds for arbitrary states $|\phi\rangle$ and $|\psi\rangle$. An operator which fulfills $\hat{A}^\dagger = \hat{A}$, is called *hermitian* and can represent an observable. An operator is *unitary* if $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{1}$. Based on these properties, proof the following statements:

a. The eigenvalues of a hermitian operator are real.

b. States corresponding to different eigenvalues of a hermitian operator must be orthogonal.

c. The absolute values of the eigenvalues of a unitary operator must be normalized.

II. MEASUREMENTS

Consider a three-state system which is initially in the quantum superposition state $|\psi\rangle = \sum_{n=1}^{3} a_n|n\rangle$, where a_n are complex numbers and the states $|n\rangle$ $(n = 1, 2, 3)$ form an orthonormal basis set of this three-dimensional Hilbert space. Such quantum mechanical system can describe for example the position of an atom in a 'triple-well' or three spectral levels of an atom. Let us introduce a 'parity' operator $\hat{S}$ which is defined by the relation $\hat{S}|n\rangle = (-1)^n|n\rangle$. Obviously the eigenvalues of this operator are $\lambda_+ = 1$ and $\lambda_- = -1$. It is important to note that the eigenvalue λ_- is twofold degenerate.

a. Write down the projection operators for the different outcomes of the measurements by the observable $\hat{S}$.

b. Which is the probability of measuring parity 1, and which is the one for measuring parity -1.

c. Give the wave function after the having observed the different values of the parity operator.

III. TIME EVOLUTION

a. Let $\hat{H}$ be a Hamiltonian with eigenvalues E_n and eigenstates $|n\rangle$. This means it exists a unitary transformation such that $\hat{H} = \hat{U}\hat{D}\hat{U}^\dagger$ with $\langle n|\hat{D}|n'\rangle = E_n\delta_{n,n'}$. Using the series expansion of exponential and the unitarity of $\hat{U}$, proof the following expression: $e^{-i\hat{H}t/\hbar} = \hat{U}e^{-i\hat{D}t/\hbar}U^\dagger$ with $\langle n|e^{-i\hat{D}t/\hbar}|n'\rangle = e^{-iE_nt/\hbar}\delta_{n,n'}$.

b. Show that if $\hat{H}$ is hermitian $e^{-i\hat{H}t/\hbar}$ is unitary. Use this result to show that time evolution in quantum mechanics is reversible.

c. (* Optional, requires a trick) Show that $e^{-i\left(\hat{H}_A+\hat{H}_B\right)t/\hbar} = e^{-i\hat{H}_At/\hbar}e^{-i\hat{H}_Bt/\hbar}e^{[\hat{H}_A,\hat{H}_B]\left(\frac{t}{\hbar}\right)^2/2}$ if $[\hat{H}_A,[\hat{H}_A,\hat{H}_B]] = [\hat{H}_B,[\hat{H}_A,\hat{H}_B]] = 0$. This means that the exponential of a sum of operators does not factorize if the operators do not commute. In the present case, the 3rd order commutators of the operators vanish, therefore we can obtain a closed expression for the exponential. It is important to note that this is not true in general.