Quantum Mechanics I, Sheet 9, Spring 2013

Responsible for this sheet: O. Peil (oleg.peil@unige.ch), office 115, Ecole de Physique

May 15, 2013 (Ecole de Physique, Auditoire Stückelberg)

Prof. D. van der Marel (dirk.vandermarel@unige.ch)
Tutorials: J. Guillod (julien.guillod@unige.ch), O. E. Peil (oleg.peil@unige.ch)

Exercises starting with (*) are optional.

I. HEISENBERG PICTURE

In Schrödinger picture, the operators are in general time-independent, as for example the position and momentum operators. The time-evolution of the system is completely captured by the quantum state $|\psi(t)\rangle$ which obeys the Schrödinger equation. However, all physical predictions of quantum mechanics are given in terms of brackets, which are invariant by performing a unitary transformation on operators and states simultaneously. The idea of Heisenberg picture is to choose this unitary transformation in such a way that the transformation of the state $|\psi(t)\rangle$ becomes time-independent; of course the transformed operators will then become time-dependent. Explicitly, a general bracket can be rewritten as

$$\langle\phi(t)|\hat{A}|\psi(t)\rangle = \langle\hat{U}(t)\phi(0)|\hat{A}|\hat{U}(t)\psi(0)\rangle = \langle\phi(0)|\hat{U}(t)^\dagger\hat{A}\hat{U}(t)|\psi(0)\rangle = \langle\phi_H|\hat{A}_H(t)|\psi_H\rangle\,,$$

where $|\psi_H\rangle = |\psi(0)\rangle$ and $|\phi_H\rangle = |\phi(0)\rangle$ are the states in Schrödinger picture and $\hat{A}_H$ is the operator A in Heisenberg picture,

$$\hat{A}_H(t) = \hat{U}(t)^\dagger\hat{A}\hat{U}(t)\,.$$

Therefore, in Heisenberg picture we obtained that states are time-independent and operators becomes time-dependent.

1. Show that the time-evolution of an operator in Heisenberg picture is given by

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_H(t) = \left[\hat{A}_H(t), \hat{H}_H(t)\right]\,,$$

where $\hat{H}_H(t)$ is the Hamiltonian operator in Heisenberg picture.

2. In one dimension, consider the Hamiltonian of a particle in a potential

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X})\,.$$

Determine the Hamiltonian in Heisenberg picture $\hat{H}_H$ in terms of the position $\hat{X}_H$ and momentum $\hat{P}_H$ operators in Heisenberg picture. Check that the commutation relations are still valid for the operators in Heisenberg picture. Write the evolution equations for the position $\hat{X}_H$ and momentum $\hat{P}_H$ operators in Heisenberg picture. Compare your result to classical mechanics.

II. ANGULAR MOMENTUM

Carbon monooxide CO is the second (after hydrogen) most abundant molecule present in the interstellar gas. This molecule is very useful for astrophysics because its rotational spectrum is rather intense and can be observed in the radio-frequency range.
To first approximation, a diatomic molecule of CO can be described as a rigid quantum rotor consisting of two point masses m_O, m_C connected with a massless rod (its length can be taken to be equal $r_b = 1.128 \times 10^{-8}$ cm). The rotation dynamics is given by a Hamiltonian

$$\hat{H}_R = \frac{\hat{J}^2 - \hat{J}_z^2}{2I},$$

where I is the molecule's moment of inertia, and $\hat{J}$ is the angular momentum.

1. Find the energies of transitions from state $|J\rangle$ to state $|J-1\rangle$.
2. Consider two sorts of molecules $^{12}C^{16}O$, $^{13}C^{16}O$ and evaluate the transition frequency $\nu_{1\to 0}$ (between states $J=1$ and $J=0$) for both of them. The following constant values can be used: proton mass $m_p = 1.67 \times 10^{-24}$g, Planck constant $\hbar = 1.055 \times 10^{-27}$ erg $\cdot$ s.
   $\Big[$*Hint:* Be careful with 2π factors; note that the spectral transition frequency is defined as $\nu = \Delta E/h \equiv \Delta E/(2\pi\hbar)$.$\Big]$

III. ROTATION OF SPIN-1/2 PARTICLES

Neutrons are prepared at $t=0$ in the state given by the superposition of the eigenvectors of the spin operator $\hat{S}_z$. Then there are injected in the experimental setup described on Figure 1 (Werner et al. Phys. Rev. Lett. 35, 1053 (1975)). We suppose that the lower beam remains a time T between $t=0$ and $t=T$ in the uniform magnetic field B_z. At point D, we measure the mean value of μ_z. What are the results in function of time T?

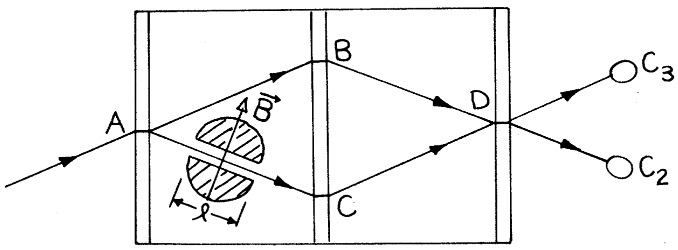

FIG. 1. A schematic diagram of the neutron interferometer. On the path AC the neutrons are in a magnetic field B (0 to 500 G) for a distance l (2 cm).

IV. (*) REVISION OF THE ONE-DIMENSIONAL SCHRÖDINGER EQUATION

A. δ-function potential

Consider a one-dimensional δ-function potential $V(x) = -g\delta(x)$. There is only one bound state ($E < 0$) in this potential. Find the energy and the wave function corresponding to this bound state.
$\Big[$*Hint:* Start off with splitting the space into three ranges $(-\infty, a)$, $[-a, a]$, (a, ∞), where a is a small number, which will eventually be put to zero.$\Big]$

B. Double δ-function potential

A double δ-function potential is defined as $V(x) = -g[\delta(x-d) + \delta(x+d)]$. The one-dimensional Schrödinger equation for a particle in this potential has two distinct solutions.

1. Using symmetry arguments prove that the two eigen wave functions correspond to the same eigenenergy.
2. Construct boundary conditions for the two wave functions and show explicitly that they result in equivalent equations for the eigenenergy.
3. Evaluate the normalization constants for the wave functions.

C. Limit of an extremely narrow square-well potential

Start out with a square-well potential,

$$V(x) = \begin{cases} -V_0, & |x| \leq a, \\ 0, & \text{otherwise}, \end{cases}$$

and show that in the limit $a \to 0$, only one bound state is left, and its energy is the same as for the δ-function potential with $g = V_0 a$. Keep in mind that to perform a well-defined limiting procedure, one has to keep the product $V_0 a$ (potential strength) constant.