Méthodes mathématiques pour physiciens I
Corrigé série 3

Echauffement. Calculer les dérivées partielles $\frac{\partial}{\partial x}$ et $\frac{\partial}{\partial y}$:

1. $\dfrac{\partial f}{\partial x}(x,y) = 1\ , \quad \dfrac{\partial f}{\partial y}(x,y) = 0\ .$
2. $\dfrac{\partial g}{\partial x}(x,y) = 0\ , \quad \dfrac{\partial g}{\partial y}(x,y) = 1\ .$
3. $\dfrac{\partial h}{\partial x}(x,y) = y\ , \quad \dfrac{\partial h}{\partial y}(x,y) = x\ .$
4. $\dfrac{\partial u}{\partial x}(x,y) = 2xy\ , \quad \dfrac{\partial u}{\partial y}(x,y) = x^2\ .$
5. $\dfrac{\partial v}{\partial x}(x,y) = \dfrac{1}{x}\ , \quad \dfrac{\partial v}{\partial y}(x,y) = -\dfrac{1}{y}\ .$

Exercice 1. Calculer les dérivées partielles $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$ et $\frac{\partial}{\partial a}$:

1. $\dfrac{\partial f}{\partial x}(x,t,a) = 3a^2x^2\sqrt{t}\ , \quad \dfrac{\partial f}{\partial t}(x,t,a) = \dfrac{a^2x^3}{2\sqrt{t}}\ , \quad \dfrac{\partial f}{\partial a}(x,t,a) = 2ax^3\sqrt{t}\ .$
2. $\dfrac{\partial g}{\partial x}(x,t,a) = \dfrac{-x}{\sqrt{a^2-x^2}}\ , \quad \dfrac{\partial g}{\partial t}(x,t,a) = 1/t\ , \quad \dfrac{\partial g}{\partial a}(x,t,a) = \dfrac{a}{\sqrt{a^2-x^2}} - 1/a\ .$
3. $\dfrac{\partial h}{\partial x}(x,t,a) = ak\sin(\omega t - kx)\ , \quad \dfrac{\partial h}{\partial t}(x,t,a) = -a\omega\sin(\omega t - kx)\ , \quad \dfrac{\partial h}{\partial a}(x,t,a) = \cos(\omega t - kx)\ .$
4. $\dfrac{\partial u}{\partial x}(x,t,a) = \dfrac{3}{5}a\ x^2(t+x^3)^{-4/5}\cos\Big(a\ (t+x^3)^{1/5}\Big)\ , \quad \dfrac{\partial u}{\partial t}(x,t,a) = \dfrac{1}{5}a\ (t+x^3)^{-4/5}\cos\Big(a\ (t+x^3)^{1/5}\Big)\ ,$
   $\dfrac{\partial u}{\partial a}(x,t,a) = (t+x^3)^{1/5}\cos\Big(a\ (t+x^3)^{1/5}\Big)\ .$
5. $\dfrac{\partial v}{\partial x}(x,t,a) = t^2a^x\ln(a)\ , \quad \dfrac{\partial v}{\partial t}(x,t,a) = 2ta^x\ , \quad \dfrac{\partial v}{\partial a}(x,t,a) = xt^2a^{x-1}\ .$

Exercice 2.

1. En calculant d'abord les dérivées partielles :
$$\partial_1 f(\mathbf{x}) = 2x_1 + 2x_2^2\ , \qquad \partial_2 f(\mathbf{x}) = 3x_2^2 + 4x_1x_2\ ,$$
et donc
$$\partial_{\mathbf{e}} f(\mathbf{x}) = e_1\partial_1 f(\mathbf{x}) + e_2\partial_2 f(\mathbf{x}) = x_1 + 2\sqrt{3}x_1x_2 + \frac{2+3\sqrt{3}}{2}x_2^2\ .$$
2. De manière similaire :
$$\partial_1 g(\mathbf{x}) = -2^{-1/2}\sin(x_1)\sin(2x_2)\ , \qquad \partial_2 g(\mathbf{x}) = 2^{1/2}\cos(x_1)\cos(2x_2)\ ,$$
et ainsi
$$\partial_{\mathbf{n}} g(\mathbf{x}) = (\mathbf{n}\cdot\nabla g)\,(\mathbf{x}) - \frac{1}{2}\sin(x_1)\sin(2x_2) + \cos(x_1)\cos(2x_2)\ .$$

Il suffit ensuite d'évaluer les dérivées partielles précédemment calculées :

3. $\partial_{\mathbf{e}} f(2,1) = \dfrac{6+11\sqrt{3}}{2}\ ,$
4. $\partial_{\mathbf{n}} g|_{x_1=0,x_2=\pi/2} = -1\ ,$
5. $\partial_{\mathbf{n}} g|_{(\pi/4,\pi/4)} = -\dfrac{\sqrt{2}}{4} = -2^{-3/2}\ .$

Exercice 3. Pour $r \neq 0$,

$$\partial_x r = \frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}\,, \qquad \partial_x \frac{1}{r} = -\frac{1}{r^2}\cdot \partial_x r = \frac{-x}{r^3}\,,$$

$$\partial_x\partial_x \frac{1}{r} = \partial_x\left(-\frac{x}{r^3}\right) = -\frac{1}{r^3} + 3\frac{x}{r^4}\partial_x r = -\frac{1}{r^3} + 3\frac{x^2}{r^5}\,,$$

et ainsi

$$\Delta\frac{1}{r} = -\frac{3}{r^3} + 3\frac{x^2+y^2+z^2}{r^5} = 0\,, \qquad \nabla\frac{1}{r} = -\frac{\mathbf{x}}{r^3}\,.$$

Exercice 4. En appliquant la règle de la dérivée en chaîne :

$$\begin{aligned}
\partial_r W(r,\theta) &= \partial_r V(x(r,\theta), y(r,\theta)) \\
&= \partial_r x\,(r,\theta)\cdot \partial_x V\,(x\,(r,\theta)\,, y\,(r,\theta)) + \partial_r y\,(r,\theta)\cdot \partial_y V\,(x\,(r,\theta)\,, y\,(r,\theta)) \\
&= \cos\theta\cdot \partial_x V\,(x\,(r,\theta)\,, y\,(r,\theta)) + \sin\theta\cdot \partial_y V\,(x\,(r,\theta)\,, y\,(r,\theta))\ , \\
\partial_\theta W(r,\theta) &= \partial_\theta V(x(r,\theta), y(r,\theta)) \\
&= \partial_\theta x\,(r,\theta)\cdot \partial_x V\,(x\,(r,\theta)\,, y\,(r,\theta)) + \partial_\theta y\,(r,\theta)\cdot \partial_y V\,(x\,(r,\theta)\,, y\,(r,\theta)) \\
&= -r\sin\theta\cdot \partial_x V\,(x\,(r,\theta)\,, y\,(r,\theta)) + r\cos\theta\cdot \partial_y V\,(x\,(r,\theta)\,, y\,(r,\theta))\ .
\end{aligned}$$

Pour simplifiant la notation il est courant d'omettre les variables :

$$\partial_r W = \cos\theta\cdot\partial_x V + \sin\theta\cdot\partial_y V\,, \qquad \partial_\theta W = r\left(-\sin\theta\cdot\partial_x V + \cos\theta\cdot\partial_y V\right)\,,$$

mais cette notation n'est pas rigoureuse car V est une fonction de x et y alors que W est une fonction de r et θ. Il faut donc garder en tête que x et y sont en fait des fonctions de r et θ. Par conséquent,

$$r\partial_x V = r\cos\theta\cdot\partial_r W - \sin\theta\cdot\partial_\theta W\,, \qquad r\partial_y V = r\sin\theta\cdot\partial_r W + \cos\theta\cdot\partial_\theta W\,,$$

et au final si $r \neq 0$,

$$\left.(\partial_x V)^2 + (\partial_y V)^2\right|_{\substack{x=r\cos\theta\\ y=r\sin\theta}} = (\partial_r W)^2 + \frac{1}{r^2}(\partial_\theta W)^2\,.$$

Exercice 5. Pour $r \neq 0$,

1. $\nabla\cdot\mathbf{x} = \partial_x x + \partial_y y + \partial_z z = 1+1+1 = 3\,,$
2. $\nabla\wedge\mathbf{x} = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\wedge\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\partial_y z - \partial_z y\\ \partial_z x - \partial_x z\\ \partial_x y - \partial_y x\end{pmatrix} = \mathbf{0}\,,$
3. $\nabla\cdot\mathbf{n} = \nabla\cdot\left(\frac{\mathbf{x}}{r}\right) = \frac{1}{r}(\nabla\cdot\mathbf{x}) + \mathbf{x}\cdot\left(\nabla\frac{1}{r}\right) = \frac{3}{r} - \mathbf{x}\cdot\frac{\mathbf{x}}{r^3} = \frac{2}{r}\,,$
4. $\nabla(\ln r) = \frac{1}{r}\nabla(r) = \frac{1}{r}\frac{\mathbf{x}}{r} = \frac{\mathbf{x}}{r^2}\,.$

Exercice 6. Pour $r \neq 0$,

1. $\nabla\cdot\mathbf{x} = \partial_x x + \partial_y y = 1+1 = 2\,,$
2. $\Delta\frac{1}{r} = \nabla\cdot\left(\nabla\frac{1}{r}\right) = -\nabla\cdot\frac{\mathbf{x}}{r^3} = -\frac{\nabla\cdot\mathbf{x}}{r^3} + \mathbf{x}\cdot\left(\nabla\frac{1}{r^3}\right) = \frac{-2}{r^3} + 3\mathbf{x}\cdot\frac{\mathbf{x}}{r^5}\,.$